

## Chapter 6

# Multivector Functions of a Multivector Variable

A. M. Moya<sup>1</sup>, V. V. Fernández<sup>1</sup> and W. A. Rodrigues Jr.<sup>1\*</sup>

<sup>1</sup> Institute of Mathematics, Statistics and Scientific Computation  
IMECC-UNICAMP CP 6065

13083-970 Campinas-SP, Brazil

<sup>2</sup> Department of Mathematical Sciences, University of Liverpool  
Liverpool, L69 3BX, UK

**ABSTRACT.** In this paper we develop with considerable details a theory of multivector functions of a  $p$ -vector variable. The concepts of limit, continuity and differentiability are rigorously studied. Several important types of derivatives for these multivector functions are introduced, as e.g., the  $A$ -directional derivative (where  $A$  is a  $p$ -vector) and the generalized concepts of curl, divergence and gradient. The derivation rules for different types of products of multivector functions and for composition of multivector functions are proved.

## 1 Introduction

This is the paper VI in the present series. Here, we develop a theory of multivector functions of a  $p$ -vector variable. For these objects we investigate with details the concepts of limit and continuity, and formulate rigorously the notion of derivation. As we will see, the concept of extensor introduced in [1] (paper II on this series) plays a crucial role in our theory of differentiability. We introduce important derivative-like operators for these multivector functions, as e.g., the  $A$ -directional derivative and the generalized concepts of curl, divergence and gradient. The derivation rules for all suitable products of multivector functions

---

Advances in Applied Clifford Algebras **11 (S3)** 79-91, *Multivector and Extensor Calculus*, (2001)

©2001 Universidad Nacional Autónoma de México. Printed in Mexico

of a  $p$ -vector variable and for composition of multivector functions are presented and proved.

## 2 Multivector Functions of a $p$ -Vector Variable

Let  $\Omega^p V$  be a subset of  $\bigwedge^p V$ . Any mapping  $F : \Omega^p V \rightarrow \bigwedge V$  will be called a *multivector function of a  $p$ -vector variable over  $V$* . In particular,  $F : \Omega^p V \rightarrow \bigwedge^q V$  is said to be a  *$q$ -vector function of a  $p$ -vector variable*, or a  *$(p, q)$ -function over  $V$* , for short. For the special cases  $q = 0$ ,  $q = 1$ ,  $q = 2, \dots$  etc. we will employ the names of *scalar, vector, bivector, ... etc. function of a  $p$ -vector variable*, respectively.

### 2.1 Limit Notion

We begin by introducing the concept of  $\delta$ -neighborhood for a multivector  $A$ .

Take any  $\delta > 0$ . The set<sup>1</sup>  $N_A(\delta) = \{X \in \bigwedge V / \|X - A\| < \delta\}$  will be called a  *$\delta$ -neighborhood of  $A$* .

The set  $N_A(\delta) - \{A\} = \{X \in \bigwedge V / 0 < \|X - A\| < \delta\}$  will be said to be a *reduced  $\delta$ -neighborhood of  $A$* .

We introduce now the concepts of cluster and interior points of  $\Omega V \subseteq \bigwedge V$ .

A multivector  $X_0 \in \bigwedge V$  is said to be a *cluster point of  $\Omega V$*  if and only if for every  $N_{X_0}(\delta) : (N_{X_0}(\delta) - \{X_0\}) \cap \Omega V \neq \emptyset$ , i.e., all reduced  $\delta$ -neighborhood of  $X_0$  contains at least one multivector of  $\Omega V$ .

A multivector  $X_0 \in \bigwedge V$  is said to be an *interior point of  $\Omega V$*  if and only if there exists  $N_{X_0}(\delta)$  such that  $N_{X_0}(\delta) \subseteq \Omega V$ , i.e., any multivector of some  $\delta$ -neighborhood of  $X_0$  belongs also to  $\Omega V$ .

It should be noted that any interior point of  $\Omega V$  is also a cluster point of  $\Omega V$ .

If the set of interior points of  $\Omega V$  coincides with  $\Omega V$ , then it is said to be an *open subset of  $\bigwedge V$* .

Take  $\Omega^p V \subseteq \bigwedge^p V$  and let  $F : \Omega^p V \rightarrow \bigwedge V$  be a multivector function of a  $p$ -vector variable and take  $X_0 \in \bigwedge^p V$  to be a cluster point of  $\Omega^p V$ .

A multivector  $M$  is said to be the *limit of  $F(X)$  for  $X$  approaching to  $X_0$*  if and only if for every real  $\varepsilon > 0$  there exists some real  $\delta > 0$  such that if  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ , then  $\|F(X) - M\| < \varepsilon$ . It is denoted by  $\lim_{X \rightarrow X_0} F(X) = M$ .

In dealing with a scalar function of a  $p$ -vector variable, say  $\Phi$ , the definition of  $\lim_{X \rightarrow X_0} \Phi(X) = \mu$  is reduced to: for every  $\varepsilon > 0$  there exists some  $\delta > 0$  such that  $|\Phi(X) - \mu| < \varepsilon$ , whenever  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ .

**Proposition 1** *Let  $F : \Omega^p V \rightarrow \bigwedge V$  and  $G : \Omega^p V \rightarrow \bigwedge V$  be two multivector functions of a  $p$ -vector variable. If there exist  $\lim_{X \rightarrow X_0} F(X)$  and  $\lim_{X \rightarrow X_0} G(X)$ , then*

<sup>1</sup>We recalls that the two double bars  $\| \cdot \|$  denotes the norm of multivectors, as defined in [2], i.e., for all  $X \in \bigwedge V : \|X\| = \sqrt{X \cdot \bar{X}}$ , where  $(\cdot)$  is any fixed euclidean scalar product.

there exists  $\lim_{X \rightarrow X_0} (F + G)(X)$  and

$$\lim_{X \rightarrow X_0} (F + G)(X) = \lim_{X \rightarrow X_0} F(X) + \lim_{X \rightarrow X_0} G(X). \quad (1)$$

**Proof.** Let  $\lim_{X \rightarrow X_0} F(X) = M_1$  and  $\lim_{X \rightarrow X_0} G(X) = M_2$ . Then, we must prove that  $\lim_{X \rightarrow X_0} (F + G)(X) = M_1 + M_2$ .

Taken a real  $\varepsilon > 0$ , since  $\lim_{X \rightarrow X_0} F(X) = M_1$  and  $\lim_{X \rightarrow X_0} G(X) = M_2$ , there must be two real numbers  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$\begin{aligned} \|F(X) - M_1\| &< \frac{\varepsilon}{2}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_1, \\ \|G(X) - M_2\| &< \frac{\varepsilon}{2}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_2. \end{aligned}$$

Thus, there is a real  $\delta = \min\{\delta_1, \delta_2\}$  such that

$$\|F(X) - M_1\| < \frac{\varepsilon}{2} \text{ and } \|G(X) - M_2\| < \frac{\varepsilon}{2},$$

for  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ . Hence, by using the triangular inequality for the norm of multivectors, it follows that

$$\begin{aligned} \|(F + G)(X) - (M_1 + M_2)\| &= \|F(X) - M_1 + G(X) - M_2\| \\ &\leq \|F(X) - M_1\| + \|G(X) - M_2\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ .

Therefore, for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ , then  $\|(F + G)(X) - (M_1 + M_2)\| < \varepsilon$ . ■

**Proposition 2** Let  $\Phi : \Omega^p V \rightarrow \mathbb{R}$  and  $F : \Omega^p V \rightarrow \wedge V$  be a scalar function and a multivector function of a  $p$ -vector variable. If there exist  $\lim_{X \rightarrow X_0} \Phi(X)$  and

$\lim_{X \rightarrow X_0} F(X)$ , then there exists  $\lim_{X \rightarrow X_0} (\Phi F)(X)$  and

$$\lim_{X \rightarrow X_0} (\Phi F)(X) = \lim_{X \rightarrow X_0} \Phi(X) \lim_{X \rightarrow X_0} F(X). \quad (2)$$

**Proof.** Let  $\lim_{X \rightarrow X_0} \Phi(X) = \Phi_0$  and  $\lim_{X \rightarrow X_0} F(X) = F_0$ . Then, we must prove that

$$\lim_{X \rightarrow X_0} (\Phi F)(X) = \Phi_0 F_0.$$

First, since  $\lim_{X \rightarrow X_0} \Phi(X) = \Phi_0$  it can be found a  $\delta_1 > 0$  such that

$$|\Phi(X) - \Phi_0| < 1, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_1,$$

i.e.,

$$|\Phi(X)| < 1 + |\Phi_0|, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_1.$$

Where the triangular inequality for real numbers  $|\alpha| - |\beta| \leq |\alpha - \beta|$  was used.

Now, taken a  $\varepsilon > 0$ , since  $\lim_{X \rightarrow X_0} \Phi(X) = \Phi_0$  and  $\lim_{X \rightarrow X_0} F(X) = F_0$ , they can be found a  $\delta_2 > 0$  and a  $\delta_3 > 0$  such that

$$|\Phi(X) - \Phi_0| < \frac{\varepsilon}{2(1 + \|F_0\|)}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_2,$$

$$\|F(X) - F_0\| < \frac{\varepsilon}{2(1 + |\Phi_0|)}, \text{ for } X \in \Omega^p V \text{ and } 0 < \|X - X_0\| < \delta_3.$$

Thus, given a real  $\varepsilon > 0$  there is a real  $\delta = \min\{\delta_1, \delta_2, \delta_3\}$  such that

$$\begin{aligned} |\Phi(X)| &< 1 + |\Phi_0|, \\ |\Phi(X) - \Phi_0| &< \frac{\varepsilon}{2(1 + \|F_0\|)}, \\ \|F(X) - F_0\| &< \frac{\varepsilon}{2(1 + |\Phi_0|)}, \end{aligned}$$

whenever  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ . Hence, using some properties of the norm of multivectors, it follows that

$$\begin{aligned} \|(\Phi F)(X) - \Phi_0 F_0\| &= \|\Phi(X)(F(X) - F_0) + (\Phi(X) - \Phi_0)F_0\| \\ &\leq |\Phi(X)| \|F(X) - F_0\| + |\Phi(X) - \Phi_0| \|F_0\| \\ &< |\Phi(X)| \|F(X) - F_0\| + |\Phi(X) - \Phi_0| (1 + \|F_0\|) \\ &< (1 + |\Phi_0|) \frac{\varepsilon}{2(1 + |\Phi_0|)} + \frac{\varepsilon}{2(1 + \|F_0\|)} (1 + \|F_0\|) = \varepsilon, \end{aligned}$$

whenever  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ .

Therefore, for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $X \in \Omega^p V$  and  $0 < \|X - X_0\| < \delta$ , then  $\|(\Phi F)(X) - \Phi_0 F_0\| < \varepsilon$ . ■

**Lemma 3** *There exists  $\lim_{X \rightarrow X_0} F(X)$  if and only if there exist either  $\lim_{X \rightarrow X_0} F^J(X)$  or  $\lim_{X \rightarrow X_0} F_J(X)$ . It holds*

$$\lim_{X \rightarrow X_0} F(X) = \sum_J \frac{1}{\nu(J)!} \lim_{X \rightarrow X_0} F^J(X) e_J = \sum_J \frac{1}{\nu(J)!} \lim_{X \rightarrow X_0} F_J(X) e^J. \quad (3)$$

**Proof.** It is an immediate consequence of eqs.(1) and (2). ■

**Proposition 4** *Let  $F : \Omega^p V \rightarrow \bigwedge V$  and  $G : \Omega^p V \rightarrow \bigwedge V$  be two multivector functions of a  $p$ -vector variable. We can define the products  $F * G : \Omega^p V \rightarrow \bigwedge V$  such that  $(F * G)(X) = F(X) * G(X)$  where  $*$  holds for either  $(\wedge)$ ,  $(\cdot)$ ,  $(\lrcorner)$  or (Clifford product). If there exist  $\lim_{X \rightarrow X_0} F(X)$  and  $\lim_{X \rightarrow X_0} G(X)$ , then there exists*

$\lim_{X \rightarrow X_0} (F * G)(X)$  and

$$\lim_{X \rightarrow X_0} (F * G)(X) = \lim_{X \rightarrow X_0} F(X) * \lim_{X \rightarrow X_0} G(X). \quad (4)$$

**Proof.** It is an immediate consequence of eq.(3). ■

## 2.2 Continuity Notion

Take  $\Omega^p V \subseteq \bigwedge^p V$ . A multivector function of a  $p$ -vector variable  $F : \Omega^p V \rightarrow \bigwedge V$  is said to be *continuous at*  $X_0 \in \Omega^p V$  if and only if there exists<sup>2</sup>  $\lim_{X \rightarrow X_0} F(X)$  and

$$\lim_{X \rightarrow X_0} F(X) = F(X_0). \quad (5)$$

**Lemma 5** *The multivector function  $X \mapsto F(X)$  is continuous at  $X_0$  if and only if any component scalar function, either  $X \mapsto F^J(X)$  or  $X \mapsto F_J(X)$  is continuous at  $X_0$ .*

**Proposition 6** *Let  $F : \Omega^p V \rightarrow \bigwedge V$  and  $G : \Omega^p V \rightarrow \bigwedge V$  be two continuous functions at  $X_0 \in \Omega^p V$ . Then, the addition  $F + G : \Omega^p V \rightarrow \bigwedge V$  such that  $(F + G)(X) = F(X) + G(X)$  and the products  $F * G : \Omega^p V \rightarrow \Omega V$  such that  $(F * G)(X) = F(X) * G(X)$ , where  $*$  means either  $(\wedge)$ ,  $(\cdot)$ ,  $(\lrcorner)$  or (Clifford product), are also continuous functions at  $X_0$ .*

**Proof.** It is an immediate consequence of eqs.(1) and (4). ■

**Proposition 7** *Let  $G : \Omega^p V \rightarrow \bigwedge^q V$  and  $F : \bigwedge^q V \rightarrow \bigwedge^r V$  be two continuous functions, the first one at  $X_0 \in \Omega^p V$  and the second one at  $G(X_0) \in \bigwedge^q V$ . Then, the composition  $F \circ G : \Omega^p V \rightarrow \bigwedge^r V$  such that  $F \circ G(X) = F(G(X))$  is a continuous function at  $X_0$ .*

## 2.3 Differentiability Notion

Let  $\Omega^p V$  be a subset of  $\bigwedge^p V$ . A  $(p, q)$ -function over  $V$ , say  $F$ , is said to be *differentiable at*  $X_0 \in \Omega^p V$  if and only if there exists a  $(p, q)$ -extensor over  $V$ , say  $f_{X_0}$ , such that

$$\lim_{X \rightarrow X_0} \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} = 0, \quad (6)$$

i.e.,

$$\lim_{H \rightarrow 0} \frac{F(X_0 + H) - F(X_0) - f_{X_0}(H)}{\|H\|} = 0. \quad (7)$$

It is remarkable that if there is such a  $(p, q)$ -extensor  $f_{X_0}$ , then it must be *unique*.

Indeed, assume that there is another  $(p, q)$ -extensor  $\widehat{f}_{X_0}$  which satisfies

$$\lim_{H \rightarrow 0} \frac{F(X_0 + H) - F(X_0) - \widehat{f}_{X_0}(H)}{\|H\|} = 0,$$

---

<sup>2</sup>Observe that  $X_0$  has to be cluster point of  $\Omega^p V$ .

or equivalently,

$$\lim_{H \rightarrow 0} \frac{\left\| F(X_0 + H) - F(X_0) - \widehat{f}_{X_0}(H) \right\|}{\|H\|} = 0.$$

By using the triangular inequality which is valid for the norm of multivectors [1], it can be easily established the following inequality

$$0 \leq \frac{\left\| f_{X_0}(H) - \widehat{f}_{X_0}(H) \right\|}{\|H\|} \leq \frac{\|F(X_0 + H) - F(X_0) - f_{X_0}(H)\|}{\|H\|} + \frac{\left\| F(X_0 + H) - F(X_0) - \widehat{f}_{X_0}(H) \right\|}{\|H\|},$$

which holds for all  $H \neq 0$  (i.e.,  $X \neq X_0$ ).

Now, taking the limits for  $H \rightarrow 0$  (i.e.,  $X \rightarrow X_0$ ) of these scalar-valued functions of a  $p$ -vector variable, we get

$$\lim_{H \rightarrow 0} \frac{\left\| f_{X_0}(H) - \widehat{f}_{X_0}(H) \right\|}{\|H\|} = 0.$$

This implies<sup>3</sup> that for every  $A \neq 0$

$$\lim_{\lambda \rightarrow 0} \frac{\left\| f_{X_0}(\lambda A) - \widehat{f}_{X_0}(\lambda A) \right\|}{\|\lambda A\|} = 0.$$

Then, it follows that for every  $A \neq 0$

$$\frac{\left\| f_{X_0}(A) - \widehat{f}_{X_0}(A) \right\|}{\|A\|} = 0,$$

i.e.,  $f_{X_0}(A) = \widehat{f}_{X_0}(A)$ . Now, for  $A = 0$  this equality trivially holds. Therefore, we have proved that  $f_{X_0} = \widehat{f}_{X_0}$ .

The  $(p, q)$ -extensor  $f_{X_0}$  will be called the *differential of the  $(p, q)$ -function  $F$  at  $X_0$* .

So that, the differentiability of  $F$  at  $X_0 \in \Omega^p V$  implies the existence of differential of  $F$  at  $X_0 \in \Omega^p V$ .

---

<sup>3</sup>In order to see that, we can use a lemma: if  $\lim_{H \rightarrow 0} \Phi(H) = 0$ , then  $\lim_{\lambda \rightarrow 0} \Phi(\lambda A) = 0$ , for all  $A \neq 0$ .

**Lemma 8** Associated to any  $(p, q)$ -function  $F$ , differentiable at  $X_0$ , there exists a  $(p, q)$ -function  $\varphi_{X_0}$ , continuous at  $X_0$ , such that

$$\varphi_{X_0}(X_0) = 0 \quad (8)$$

and for every  $X \in \Omega^p V$  it holds

$$F(X) = F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \varphi_{X_0}(X). \quad (9)$$

**Proof.** Since the  $(p, q)$ -function  $F$  is differentiable at  $X_0$ , we can define an auxiliary  $(p, q)$ -function  $\varphi_{X_0}$  by

$$\varphi_{X_0}(X) = \begin{cases} 0 & \text{for } X = X_0 \\ \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} & \text{for } X \neq X_0 \end{cases}.$$

It satisfies  $\varphi_{X_0}(X_0) = 0$  and, by taking limit of  $\varphi_{X_0}(X)$  for  $X \rightarrow X_0$ , we have

$$\lim_{X \rightarrow X_0} \varphi_{X_0}(X) = \lim_{X \rightarrow X_0} \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

It follows that  $\varphi_{X_0}$  is continuous at  $X_0$  and  $\varphi_{X_0}(X_0) = 0$ .

Recall now that for  $X \neq X_0$  it follows the multivector identity

$$F(X) = F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \varphi_{X_0}(X),$$

which for  $X = X_0$  it is trivially true. ■

As happens in the  $\mathbb{R}^n$  calculus, differentiability implies continuity. Indeed, by taking limits for  $X \rightarrow X_0$  on both sides of eq.(9), we get  $\lim_{X \rightarrow X_0} F(X) = F(X_0)$ .

### 2.3.1 Directional Derivative

Since  $\Omega^p V$  is an open subset of  $\wedge^p V$ , any  $p$ -vector  $X_0$  belonging to  $\Omega^p V$  is an interior point of  $\Omega^p V$ , i.e., there is some  $\varepsilon$ -neighborhood of  $X_0$ , say  $N_{X_0}^p(\varepsilon)$ , such that  $N_{X_0}^p(\varepsilon) \subseteq \Omega^p V$ .

Now, take a non-zero  $p$ -vector  $A$  and choose a real number  $\lambda$  such that  $0 < |\lambda| < \frac{\varepsilon}{\|A\|}$ . Then, from the obvious inequality  $\|(X_0 + \lambda A) - X_0\| = |\lambda| \|A\| < \varepsilon$  it follows that  $(X_0 + \lambda A) \in N_{X_0}^p(\varepsilon)$ . Thus,  $(X_0 + \lambda A) \in \Omega^p V$ .

There exists  $\lim_{\lambda \rightarrow 0} \frac{F(X_0 + \lambda A) - F(X_0)}{\lambda}$  and it equals  $f_{X_0}(A)$ .

Indeed, by using eq.(9) we have

$$\begin{aligned} \frac{F(X_0 + \lambda A) - F(X_0)}{\lambda} &= \frac{f_{X_0}(\lambda A) + \|\lambda A\| \varphi_{X_0}(X_0 + \lambda A)}{\lambda} \\ &= f_{X_0}(A) \pm \|A\| \varphi_{X_0}(X_0 + \lambda A). \end{aligned}$$

Now, by taking limits for  $\lambda \rightarrow 0$  on these  $q$ -vector functions of a real variable, using the continuity of  $\varphi_{X_0}$  at  $X_0$  and eq.(8), the required result follows.

The  $A$ -directional derivative of  $F$  at  $X_0$ , conveniently denoted by  $F'_A(X_0)$ , is defined to be

$$F'_A(X_0) = \lim_{\lambda \rightarrow 0} \frac{F(X_0 + \lambda A) - F(X_0)}{\lambda}, \quad (10)$$

i.e.,

$$F'_A(X_0) = \left. \frac{d}{d\lambda} F(X_0 + \lambda A) \right|_{\lambda=0}. \quad (11)$$

The above observation yields a noticeable multivector identity,

$$F'_A(X_0) = f_{X_0}(A) \quad (12)$$

which relates the  $A$ -directional derivation with the differentiation.

Hence, because of the linearity property for  $(p, q)$ -extensors it follows that the  $A$ -directional derivative of  $F$  at  $X_0$  has the remarkable property: for any  $\alpha, \beta \in \mathbb{R}$  and  $A, B \in \bigwedge^p V$

$$F'_{\alpha A + \beta B}(X_0) = \alpha F'_A(X_0) + \beta F'_B(X_0). \quad (13)$$

**Proposition 9** *Let  $X : S \rightarrow \Lambda^q V$  be any  $q$ -vector function of a real variable derivable at  $\lambda_0 \in S$ . Then,  $X$  is differentiable at  $\lambda_0$  and the differential of  $X$  at  $\lambda_0$  is  $X_{\lambda_0} \in \text{ext}_0^q(V)$  given by*

$$X_{\lambda_0}(\alpha) = X'(\lambda_0)\alpha, \quad (14)$$

where  $X'(\lambda_0)$  is the derivative of  $X$  at  $\lambda_0$ .

**Proof.** We only need to prove that

$$\lim_{\lambda \rightarrow \lambda_0} \frac{X(\lambda) - X(\lambda_0) - X'(\lambda_0)(\lambda - \lambda_0)}{|\lambda - \lambda_0|} = 0.$$

Since  $X$  is derivable at  $\lambda_0$ , there is a  $q$ -vector function of real variable, say  $\xi_{\lambda_0}$ , continuous at  $\lambda_0$  such that for all  $\lambda \in S$

$$X(\lambda) = X(\lambda_0) + (\lambda - \lambda_0)X'(\lambda_0) + (\lambda - \lambda_0)\xi_{\lambda_0}(\lambda),$$

where  $\xi_{\lambda_0}(\lambda_0) = 0$ .

Hence, it follows that for all  $\lambda \neq \lambda_0$

$$\frac{X(\lambda) - X(\lambda_0) - X'(\lambda_0)(\lambda - \lambda_0)}{|\lambda - \lambda_0|} = \pm \xi_{\lambda_0}(\lambda).$$

Thus, by taking limits for  $\lambda \rightarrow \lambda_0$  on both sides, we get the expected result. ■

From eqs.(12) and (14), it should be noted that the  $\alpha$ -directional derivative of  $X$  at  $\lambda_0$  is given by

$$X'_\alpha(\lambda_0) = X'(\lambda_0)\alpha. \quad (15)$$

### 2.3.2 Differentiation Rules

Take two open subset of  $\bigwedge^p V$ , say  $\Omega_1^p V$  and  $\Omega_2^p V$ , such that  $\Omega_1^p V \cap \Omega_2^p V \neq \emptyset$ .

**Theorem 10** Let  $F : \Omega_1^p V \rightarrow \bigwedge^q V$  and  $G : \Omega_2^p V \rightarrow \bigwedge^q V$  be two differentiable functions at  $X_0 \in \Omega_1^p V \cap \Omega_2^p V$ . Denote the differentials of  $F$  and  $G$  at  $X_0$  by  $f_{X_0}$  and  $g_{X_0}$ , respectively.

The addition  $F + G : \Omega_1^p V \cap \Omega_2^p V \rightarrow \bigwedge^q V$  such that  $(F + G)(X) = F(X) + G(X)$  and the products  $F * G : \Omega_1^p V \cap \Omega_2^p V \rightarrow \bigwedge^q V$  such that  $(F * G)(X) = F(X) * G(X)$ , where  $*$  means either  $(\wedge)$ ,  $(\cdot)$ ,  $(\perp)$  or (Clifford product), are also differentiable functions at  $X_0$ .

The differential of  $F + G$  at  $X_0$  is  $f_{X_0} + g_{X_0}$  and the differentials of  $F * G$  at  $X_0$  are given by  $A \mapsto f_{X_0}(A) * G(X_0) + F(X_0) * g_{X_0}(A)$ .

**Proof.** We must prove that  $s_{X_0} = f_{X_0} + g_{X_0} \in \text{ext}_p^q(V)$  satisfies

$$\lim_{X \rightarrow X_0} \frac{(F + G)(X) - (F + G)(X_0) - s_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

And,  $p_{X_0}(A) = f_{X_0}(A) * G(X_0) + F(X_0) * g_{X_0}(A)$ , more general extensors over  $V$ , verify

$$\lim_{X \rightarrow X_0} \frac{(F * G)(X) - (F * G)(X_0) - p_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

Since  $F$  and  $G$  are differentiable at  $X_0$ , there are two  $(p, q)$ -functions  $\varphi_{X_0}$  and  $\psi_{X_0}$ , continuous at  $X_0$ , such that for all  $X \in \Omega_1^p V \cap \Omega_2^p V$

$$\begin{aligned} F(X) &= F(X_0) + f_{X_0}(X - X_0) + \|X - X_0\| \varphi_{X_0}(X), \\ G(X) &= G(X_0) + g_{X_0}(X - X_0) + \|X - X_0\| \psi_{X_0}(X), \end{aligned}$$

where  $\varphi_{X_0}(X_0) = \psi_{X_0}(X_0) = 0$ .

Hence, the following multivector identities which hold for all  $X \neq X_0$  can be easily deduced

$$\frac{(F + G)(X) - (F + G)(X_0) - s_{X_0}(X - X_0)}{\|X - X_0\|} = \varphi_{X_0}(X) + \psi_{X_0}(X)$$

and

$$\begin{aligned} & \frac{(F * G)(X) - (F * G)(X_0) - p_{X_0}(X - X_0)}{\|X - X_0\|} \\ = & \varphi_{X_0}(X) * G(X_0) + F(X_0) * \psi_{X_0}(X) \\ & + \varphi_{X_0}(X) * g_{X_0}(X - X_0) + f_{X_0}(X - X_0) * \psi_{X_0}(X) \\ & + \frac{f_{X_0}(X - X_0) * g_{X_0}(X - X_0)}{\|X - X_0\|} + \|X - X_0\| \varphi_{X_0}(X) * \psi_{X_0}(X). \end{aligned}$$

Now, by taking limits for  $X \rightarrow X_0$  on both sides of these multivector identities<sup>4</sup>, we get the required results. ■

<sup>4</sup>For calculating some limits we have used a useful lemma. For any  $f \in \text{ext}_p^q(V)$  there exists a real number  $M \geq 0$  such that for every  $X \in \bigwedge^p V : \|f(X)\| \leq M \|X\|$ .

In accordance with eq.(12) all differentiation rule turns out to be an  $A$ -directional derivation rule.

For the addition of two differentiable functions  $F$  and  $G$  we have

$$(F + G)'_A(X_0) = (f_{X_0} + g_{X_0})(A) = f_{X_0}(A) + g_{X_0}(A),$$

i.e.,

$$(F + G)'_A(X_0) = F'_A(X_0) + G'_A(X_0). \quad (16)$$

For the products  $F * G$  we get

$$(F * G)'_A(X_0) = f_{X_0}(A) * G(X_0) + F(X_0) * g_{X_0}(A),$$

i.e.,

$$(F * G)'_A(X_0) = F'_A(X_0) * G(X_0) + F(X_0) * G'_A(X_0). \quad (17)$$

**Theorem 11** *Take an open subset of  $\bigwedge^p V$ , say  $\Omega^p V$ . Let  $G : \Omega^p V \rightarrow \bigwedge^q V$  and  $F : \bigwedge^q V \rightarrow \bigwedge^r V$  be two differentiable functions, the first one at  $X_0 \in \Omega^p V$  and the second one at  $G(X_0) \in \bigwedge^q V$ . Denote by  $g_{X_0}$  and  $f_{G(X_0)}$  the differentials of  $G$  at  $X_0$  and of  $F$  at  $G(X_0)$ , respectively.*

*The composition  $F \circ G : \Omega^p V \rightarrow \bigwedge^r V$  such that  $F \circ G(X) = F(G(X))$  is also a differentiable function at  $X_0$ . The differential of  $F \circ G$  at  $X_0$  is  $f_{G(X_0)} \circ g_{X_0}$ .*

**Proof.** We must prove that for  $f_{G(X_0)} \circ g_{X_0} \in \text{ext}_p^r(V)$  it holds

$$\lim_{X \rightarrow X_0} \frac{F \circ G(X) - F \circ G(X_0) - f_{G(X_0)} \circ g_{X_0}(X - X_0)}{\|X - X_0\|} = 0.$$

Since  $G$  is differentiable at  $X_0$  and  $F$  is differentiable at  $G(X_0)$ , there are a  $(p, q)$ -function  $X \mapsto \psi_{X_0}(X)$  and a  $(q, r)$ -function  $Y \mapsto \varphi_{G(X_0)}(Y)$ , the first one continuous at  $X_0$  and the second one continuous at  $G(X_0)$ , such that for all  $X \in \Omega^p V$  and  $Y \in \bigwedge^q V$

$$\begin{aligned} G(X) &= G(X_0) + g_{X_0}(X - X_0) + \|X - X_0\| \psi_{X_0}(X), \\ F(Y) &= F(G(X_0)) + f_{G(X_0)}(Y - G(X_0)) + \|Y - G(X_0)\| \varphi_{G(X_0)}(Y), \end{aligned}$$

where  $\psi_{X_0}(X_0) = 0$  and  $\varphi_{G(X_0)}(G(X_0)) = 0$ .

Hence, it follows easily a multivector identity which holds for all  $X \neq X_0$

$$\begin{aligned} &\frac{F \circ G(X) - F \circ G(X_0) - f_{G(X_0)} \circ g_{X_0}(X - X_0)}{\|X - X_0\|} \\ &= f_{G(X_0)} \circ \psi_{X_0}(X) + \frac{\|G(X) - G(X_0)\|}{\|X - X_0\|} \varphi_{G(X_0)} \circ G(X). \end{aligned}$$

Now, by taking limits for  $X \rightarrow X_0$  on both sides, using the equations:  $\lim_{X \rightarrow X_0} f_{G(X_0)} \circ \psi_{X_0}(X) = 0$  and  $\lim_{X \rightarrow X_0} \frac{\|G(X) - G(X_0)\|}{\|X - X_0\|} \varphi_{G(X_0)} \circ G(X) = 0$ , we get the expected result. ■

This *chain rule* for differentiation turns out to be a chain rule for  $A$ -directional derivation.

For a differentiable  $G$  at  $X_0$  and a differentiable  $F$  at  $G(X_0)$  we have

$$\begin{aligned} (F \circ G)'_A(X_0) &= f_{G(X_0)} \circ g_{X_0}(A) = f_{G(X_0)}(G'_A(X_0)), \\ (F \circ G)'_A(X_0) &= F'_{G'_A(X_0)}(G(X_0)). \end{aligned} \quad (18)$$

We study now two very important particular cases of the general chain rule for the  $A$ -directional derivation:

For  $p > 0$ ,  $q = 0$  and  $r > 0$ , i.e., for the  $A$ -directional derivative of the composition of  $\Phi : \Omega^p V \rightarrow \mathbb{R}$  with  $X : \mathbb{R} \rightarrow \bigwedge^r V$  at  $X_0 \in \Omega^p V$ , by using eq.(18) and eq.(15), we have

$$\begin{aligned} (X \circ \Phi)'_A(X_0) &= X'_{\Phi'_A(X_0)}(\Phi(X_0)), \\ (X \circ \Phi)'_A(X_0) &= X'(\Phi(X_0))\Phi'_A(X_0). \end{aligned} \quad (19)$$

For  $p = 0$ ,  $q > 0$  and  $r > 0$ , i.e., for the derivative of the composition of  $X : S \rightarrow \bigwedge^q V$  with  $F : \bigwedge^q V \rightarrow \bigwedge^r V$  at  $\lambda_0 \in S$ , by using eq.(15), eq.(18) and eq.(13), we have

$$\begin{aligned} (F \circ X)'(\lambda_0)\alpha &= (F \circ X)'_\alpha(\lambda_0) = F'_{X'_\alpha(\lambda_0)}(X(\lambda_0)), \\ (F \circ X)'(\lambda_0)\alpha &= F'_{X'(\lambda_0)\alpha}(X(\lambda_0)). \end{aligned} \quad (20)$$

### 2.3.2 Derivatives

Let  $(\{e_k\}, \{e^k\})$  be a pair of reciprocal bases of  $V$ . Let  $F : \Omega^p V \rightarrow \bigwedge^q V$  be any differentiable function at  $X_0 \in \Omega^p V$ . Define the set  $\mathbf{\Lambda}^p V = \{X \in \Omega^p V / F \text{ is differentiable at } X\} \subseteq \Omega^p V$ .

It follows that it must exist a well-defined function  $F'_A : \mathbf{\Lambda}^p V \rightarrow \bigwedge^q V$  such that  $F'_A(X)$  equals the  $A$ -directional derivative of  $F$  at each  $X \in \mathbf{\Lambda}^p V$ . It is called the *A-directional derivative function of F*.

Then, we can define exactly *four* derivative-like functions for  $F$ , namely,  $*F' : \mathbf{\Lambda}^p V \rightarrow \bigwedge^q V$  such that

$$\begin{aligned} *F'(X) &= \frac{1}{p!}(e^{j_1} \wedge \dots \wedge e^{j_p}) * F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X) \\ &= \frac{1}{p!}(e_{j_1} \wedge \dots \wedge e_{j_p}) * F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X), \end{aligned} \quad (21)$$

where  $*$  means either  $(\wedge)$ ,  $(\cdot)$ ,  $(\lrcorner)$  or (*Clifford product*).

Whichever  $*F'$  is a well-defined function associated to  $F$ , since  $*F'(X)$  are multivectors which do not depend on the choice of  $(\{e_k\}, \{e^k\})$ .

We will call  $\wedge F'$ ,  $\cdot F'$ ,  $\lrcorner F'$  and  $F'$  (i.e.,  $*$   $\equiv$  (*Clifford product*)) respectively the (generalized) *curl*, *scalar divergence*, *left contracted divergence* and *gradient of F*. Sometimes the gradient of  $F$  is called the *standard derivative of F*.

On the real vector space of differentiable  $(p, q)$ -functions over  $V$  we can introduce exactly four derivative-like operators, namely,  $F \mapsto \partial_X * F$  such that

$$\partial_X * F = *F', \quad (22)$$

i.e., for every  $X \in \Lambda^p V$

$$\begin{aligned} \partial_X * F(X) &= \frac{1}{p!} (e^{j_1} \wedge \dots \wedge e^{j_p}) * F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X) \\ &= \frac{1}{p!} (e_{j_1} \wedge \dots \wedge e_{j_p}) * F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X). \end{aligned} \quad (23)$$

The special cases  $\partial_X \wedge$ ,  $\partial_X \cdot$ ,  $\partial_X \lrcorner$  and  $\partial_X$  (i.e.,  $*$   $\equiv$  (Clifford product)) will be called respectively the (generalized) *curl*, *scalar divergence*, *left contracted divergence* and *gradient operator*.

For differentiable functions it is also possible to introduce a remarkable operator denoted by  $A \cdot \partial_X$ , and defined as follows

$$\begin{aligned} A \cdot \partial_X F(X) &= (A \cdot \frac{1}{p!} e^{j_1} \wedge \dots \wedge e^{j_p}) F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X), \\ &= (A \cdot \frac{1}{p!} e_{j_1} \wedge \dots \wedge e_{j_p}) F'_{e^{j_1} \wedge \dots \wedge e^{j_p}}(X), \end{aligned} \quad (24)$$

i.e., by eq.(13)

$$A \cdot \partial_X F(X) = F'_A(X). \quad (25)$$

The operator  $A \cdot \partial_X$  is called the  $A$ -directional derivative operator. It maps  $F \rightarrow F'_A$ , i.e.,  $A \cdot \partial_X F = F'_A$ .

Now, we write out the property expressed by eq.(13) using the operator  $A \cdot \partial_X$ . We have,

$$(\alpha A + \beta B) \cdot \partial_X F(X_0) = \alpha A \cdot \partial_X F(X_0) + \beta B \cdot \partial_X F(X_0). \quad (26)$$

We have then a suggestive operator identity

$$(\alpha A + \beta B) \cdot \partial_X = \alpha A \cdot \partial_X + \beta B \cdot \partial_X. \quad (27)$$

We have also rules holding for the  $A$ -directional derivation of addition, products and composition of differentiable functions, and eq.(16), eq.(17) and eq.(18) can be written as:

$$A \cdot \partial_X (F + G)(X_0) = A \cdot \partial_X F(X_0) + A \cdot \partial_X G(X_0), \quad (28)$$

i.e.,  $A \cdot \partial_X (F + G) = A \cdot \partial_X F + A \cdot \partial_X G$ .

$$A \cdot \partial_X (F * G)(X_0) = A \cdot \partial_X F(X_0) * G(X_0) + F(X_0) * A \cdot \partial_X G(X_0), \quad (29)$$

i.e.,  $A \cdot \partial_X (F * G) = (A \cdot \partial_X F) * G + F * (A \cdot \partial_X G)$ .

If  $X \mapsto G(X)$  and  $Y \mapsto F(Y)$ , then

$$A \cdot \partial_X (F \circ G)(X_0) = A \cdot \partial_X G(X_0) \cdot \partial_Y F(G(X_0)). \quad (30)$$

### 3 Conclusions

We studied in detail the concepts of limit, continuity and differentiability for multivector functions of a  $p$ -vector variable. Several types of derivatives for these objects have been introduced as, e.g., the  $A$ -directional derivative and the generalized curl, divergence and gradient. We saw that the concept of extensor plays a key role in the formulation of the notion of differentiability, it implies the existence of the differential extensor. We have proved the basic derivation rules for all suitable products of multivector functions and for composition of multivector functions. The generalization of these results towards a formulation of a general theory of multivector functions of several multivector variables can be done easily. The concept of multivector derivatives has been first introduced in [3]. We think that our presentation is an improvement of that presentation, clearing many issues.

In the following paper about multivector *functionals*, we will see that the gradient-derivative plays a key role in the formulation of derivation concepts for the so-called induced multivector functionals.

**Acknowledgement:** V. V. Fernández is grateful to FAPESP for a postdoctoral fellowship. W. A. Rodrigues Jr. is grateful to CNPq for a senior research fellowship (contract 201560/82-8) and to the Department of Mathematics of the University of Liverpool for the hospitality. Authors are also grateful to Drs. P. Lounesto, I. Porteous, and J. Vaz, Jr. for their interest on our research and useful discussions.

### Bibliography

- [1] Fernández, V. V. Moya, A. M., and Rodrigues, W. A. Jr., Extensors (paper II of a series of seven), this issue of *AACA* **11**(S3), (2001).
- [2] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Multivector Functions of a Real Variable (paper V of a series of seven), *AACA* **11**(S3), (2001)
- [3] Hestenes, D. and Sobczyk, G., “Clifford Algebra to Geometric Calculus”, Reidel Publ. Co., Dordrecht, 1984.