A CONSTRUCTIVE DEFINITION FOR BILINEAR FORMS ON SPINORS

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Abstract. The following article gives a constructive definition of a bilinear form on a space of spinors. The construction is purely algebraic terms without reference to a matrix basis. The new definition is of interest both in principle, indicating how structures of simple algebras are inherited by their minimal ideals, and for the practical implementation of a symbolic yet efficient way of implementing Clifford algebras in a computer algebra system as is shown in [1]. An analogous construction for the charge conjugation operation is indicated. Explicit connections to the standard definition and development of bilinear forms on spinors are made throughout.

1. Introduction

This article grew out of the desire to lift the mystery enshrouding an enigmatic Clifford element C that happens to show up in the definition of bilinear forms on spinors in [p. 63 ref. 2].[1] (This is one of the cases where an existence theorem is nice as long as you do not want to do an actual computation.) The beauty of a representation independent, algebraic approach to spinors seemed to be spoiled, since one supposedly needs a matrix basis to get one's hands on C. In the construction of bilinear forms presented here, as well as in a similar construction of the charge conjugation operation which is based on the same principles, this beauty is restored.

Apart from this noble endeavor, the constructions are very useful for implementing bilinear forms and charge conjugation on spinors in computer algebra packages in a purely symbolic yet efficient manner.

[1] The element in the reference is actually named J, but there are already too many j's in this paper.

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Section 2 contains some introductory material and presents the usual definition to bilinear forms on spinors. Section 3 presents our constructive definition and compares it to the previous one. Section 4 gives an example application of the new approach. Section 5 formulates the principle of inheritance of structures from the simple algebra to its minimal ideals which underlies the constructive approach and is also applicable to charge conjugation. Section 6 summarizes our results.

2. Bilinear Forms on Spinors

A space of spinors in a Clifford algebra $\mathfrak{A}$ is given by $\mathfrak{A} \vee \mathbb{P}$ where $\mathbb{P}$ is a primitive idempotent. (For references on Clifford algebras see besides [2] and [3]). We will assume that $\mathfrak{A}$ is simple so that it is isomorphic to the tensor product of a division algebra $\mathfrak{D}$ and a total matrix algebra $\mathfrak{M}$:

$$\mathfrak{A} \cong \mathfrak{D} \otimes \mathfrak{M}. \tag{1}$$

(We are concerned with the case where $\mathfrak{D}$ is a division algebra over $\mathbb{R}$ even though the construction generalizes.) For $\mathbb{P}$ to be primitive it is necessary and sufficient that $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}$ is isomorphic to $\mathfrak{D}$ as an algebra:

$$\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P} \cong \mathfrak{D}. \tag{2}$$

(Note that we denote multiplication in $\mathfrak{A}$ by "$\vee$". Multiplication by elements of $\mathfrak{D}$ is often denoted just by juxtaposition especially when it is helpful to view the expression as element of a $\mathfrak{D}$-module.)

Given an involution $\mathcal{J}$ and a primitive idempotent P, a non-degenerate bilinear form on spinors can be defined by the map

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{A} \vee \mathbb{P} \times \mathfrak{A} \vee \mathbb{P} &\to \mathbb{P} \vee \mathfrak{A} \vee \mathbb{P} \equiv \mathfrak{D} \\ (\phi, \psi) \quad &\mapsto \langle \phi, \psi \rangle := C^{-1} \vee \phi^{\mathcal{J}} \vee \psi \,, \end{aligned} \tag{3}$$

where C has the property

$$\mathbb{P}^{\mathcal{J}} = C \vee \mathbb{P} \vee C^{-1}. \tag{4}$$

(As indicated by the notation in (3), we identify $\mathfrak{D}$ with $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}$ from now on. Of course, there is an embedding $\mathfrak{D} \hookrightarrow \mathfrak{A}$ induced by the isomorphism (1), which is compatible with this identification.) Since any two primitive idempotents of a simple algebra are equivalent, such an element C always exists. We will see later (cf. Eq. (10)) that bilinear forms for different such elements only differ by a factor in $\mathfrak{D}$. Note that this definition is a one which demonstrates the existence of bilinear forms, but it does not indicate how the element C is to obtained in practice. Indeed, in order to calculate C one usually takes recurse to a matrix representation of $\mathfrak{A}$. Therefore one can no longer choose to work purely with the symbolic algebra $\mathfrak{A}$ and one needs to specify P.

The bilinear form defined in Eq. (3) is invariant under the left action of the spin group,

$$\langle s\phi, s\psi\rangle = C^{-1} \vee \phi^{\mathcal{J}} \vee s^{\mathcal{J}} \vee s \vee \psi = \langle \phi, \psi\rangle, \tag{5}$$

if $s^{\mathcal{J}} \vee s = 1$ for all elements of the spin group. This is true if $\mathcal{J}$ is one of the involutions denoted by ξ, $\xi\eta$, ξ^* and $\xi\eta^*$ in [2]. (Other involutions equivalent to these induce bilinear forms which are invariant under a conjugated action of the spin group.) C and $\mathcal{J}$ induce an anti-automorphism j_C on $\mathfrak{D}$:

$$\begin{aligned} j_C : I\!\!P \vee \mathfrak{A} \vee I\!\!P \equiv \mathfrak{D} &\rightarrow I\!\!P \vee \mathfrak{A} \vee I\!\!P \equiv \mathfrak{D} \\ \delta \qquad &\mapsto C^{-1} \vee \delta^{\mathcal{J}} \vee C, \end{aligned} \tag{6}$$

For j_C to be an involution we need $C^{-1} \vee C^{\mathcal{J}}$ to lie in the center of $\mathfrak{D}$. Actually, C can be rescaled by an element of $\mathfrak{D}$ to satisfy

$$C^{\mathcal{J}} = \zeta\, C, \qquad \zeta \in \{+1, -1\}, \tag{7}$$

where the sign may be chosen arbitrarily if the original j_C was a non-trivial anti-automorphism. (This means that the sign is fixed if $\mathfrak{D} = \mathbb{R}$, since there are no non-trivial anti-automorphisms of $\mathbb{R}$, whereas for $\mathfrak{D} = \mathbb{C}$, the sign is only fixed if $\mathcal{J}$ does not "involve" complex conjugation, while for the case of the quaternions, $\mathfrak{D} = \mathbb{H}$, all anti-automorphisms of $\mathfrak{D} \otimes \mathfrak{M}$ act non-trivially on $\mathfrak{D}$.) The symmetry factor ζ describes the symmetry properties of the bilinear form, i.e., whether it is symmetric or symplectic, hermitian or antihermitian:

$$\begin{aligned} \langle \phi, \psi \rangle &= C^{-1} \vee \phi^{\mathcal{J}} \vee \psi = \zeta\, C^{-1} \vee \phi^{\mathcal{J}} \vee \psi \vee C^{-1\mathcal{J}} \vee C \\ &= \zeta\, C^{-1} \vee (C^{-1} \vee \psi^{\mathcal{J}} \vee \phi)^{\mathcal{J}} \vee C \\ &= \zeta\, \langle \psi, \phi \rangle^{j_C}. \end{aligned} \tag{8}$$

3. A Constructive Definition of Bilinear Forms

There are many invertible elements $C \in \mathfrak{A}$ that satisfy Eq. (4) but the bilinear forms arising from those only differ by a factor in $\mathfrak{D}$. This observation should raise some suspicion about how much information about C actually enters into the definition of the bilinear form in Eq. (3). In the approach presented in this section we suggest that the $\mathfrak{D}$-module structure of $I\!\!P \vee \mathfrak{A} \vee I\!\!P^{\mathcal{J}}$ provides a both simpler and more fundamental avenue to introduce bilinear forms than the one above. Of course, both approaches are completely equivalent.

We examine the role of C^{-1} in $\langle\phi,\psi\rangle$ (see Eq. (3)). $\psi \in \mathfrak{A}_\vee I\!P$ implies $\psi^{\mathcal{J}} \in I\!P^{\mathcal{J}}{}_\vee \mathfrak{A}$, which allows us to replace $\psi^{\mathcal{J}}$ by $I\!P^{\mathcal{J}}{}_\vee \psi^{\mathcal{J}}$. Since $\langle\phi,\psi\rangle \in I\!P_\vee \mathfrak{A}_\vee I\!P$ we may multiply by $I\!P$ from the left, whence

$$\langle\phi,\psi\rangle = I\!P_\vee C^{-1}{}_\vee I\!P^{\mathcal{J}}{}_\vee \phi^{\mathcal{J}}{}_\vee \psi. \tag{9}$$

So once we know $I\!P_\vee C^{-1}{}_\vee I\!P^{\mathcal{J}} \in I\!P_\vee \mathfrak{A}_\vee I\!P^{\mathcal{J}}$, the bilinear form is completely determined. However, by Eq. (4)

$$I\!P_\vee \mathfrak{A}_\vee I\!P^{\mathcal{J}} = I\!P_\vee \mathfrak{A}_\vee C_\vee I\!P_\vee C^{-1} = \mathfrak{D}_\vee C^{-1} \tag{10}$$

is a 1-dimensional left $\mathfrak{D}$-module. Therefore, there is a simple, concise, constructive definition of bilinear forms. Namely any non-zero element $W \in I\!P_\vee \mathfrak{A}_\vee I\!P^{\mathcal{J}}$

$$\langle\phi,\psi\rangle_W := W_\vee \phi^{\mathcal{J}}{}_\vee \psi = \rho\, I\!P_\vee C^{-1}{}_\vee I\!P^{\mathcal{J}}{}_\vee \phi^{\mathcal{J}}{}_\vee \psi = \rho\,\langle\phi,\psi\rangle \tag{11}$$

defines a non-degenerate bilinear form that agrees with $\langle\cdot,\cdot\rangle$ up to a factor in $\rho \in \mathfrak{D}$, which is determined by $W = \rho\, I\!P_\vee C^{-1}{}_\vee I\!P^{\mathcal{J}}$, and all bilinear forms derived from $\mathcal{J}$ can be obtained in this fashion. Note that this definition eliminated any redundant information contained in C. While different C's satisfying Eq. (4) may define the same bilinear form, there is a 1-1 correspondence between elements of $I\!P_\vee \mathfrak{A}_\vee I\!P^{\mathcal{J}}$ and bilinear forms.

Furthermore the symmetry of the bilinear form defined in Eq. (11) depends on the anti-automorphism j_W on $\mathfrak{D}$ induced by W and $\mathcal{J}$:

$$\delta^{j_W} W = W_\vee \delta^{\mathcal{J}}. \tag{12}$$

Since the left $\mathfrak{D}$-module $I\!P_\vee \mathfrak{A}_\vee I\!P^{\mathcal{J}}$ is obviously stable under $\mathcal{J}$, we have $\theta \in \mathfrak{D}$ such that

$$W^{\mathcal{J}} = \theta\, W. \tag{13}$$

With this definition we see that j_W is an involution if θ is in the center of $\mathfrak{D}$:

$$\begin{aligned}(\delta^{j_W})^{j_W} W &= W_\vee (\delta^{j_W})^{\mathcal{J}} = (\delta^{j_W}{}_\vee W^{\mathcal{J}})^{\mathcal{J}} = (\delta^{j_W}{}_\vee W_\vee (\theta^{-1})^{\mathcal{J}})^{\mathcal{J}} \\ &= (W_\vee \delta^{\mathcal{J}}{}_\vee (\theta^{-1})^{\mathcal{J}})^{\mathcal{J}} = \theta^{-1}\delta\, W^{\mathcal{J}} \\ &= \theta^{-1}\delta\,\theta\, W. \end{aligned} \tag{14}$$

The behavior of the bilinear form $\langle\cdot,\cdot\rangle_W$ under exchange of arguments can be expressed in terms of j_W and θ:

$$\begin{aligned}\langle\phi,\psi\rangle_W W^{\mathcal{J}} &= W_\vee \phi^{\mathcal{J}}{}_\vee \psi_\vee W^{\mathcal{J}} = W_\vee (W_\vee \psi^{\mathcal{J}}{}_\vee \phi)^{\mathcal{J}} \\ &= W_\vee (\langle\psi,\phi\rangle_W)^{\mathcal{J}} = \langle\psi,\phi\rangle_W^{j_W} W \\ &= \langle\psi,\phi\rangle_W^{j_W}\theta^{-1} W^{\mathcal{J}} \\ \Longrightarrow \quad \langle\phi,\psi\rangle_W &= \langle\psi,\phi\rangle_W^{j_W}\theta^{-1} \end{aligned} \tag{15}$$

Therefore, choosing $W \in \mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ such that $W^{\mathcal{J}} = \pm W$ induces products of the corresponding symmetry. It is actually simple to find an appropriate W. If $\mathfrak{D}$ is a division algebra over the reals, the left $\mathfrak{D}$-module $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ is in particular a vector space over $\mathbb{R}$ with $\mathcal{J}$ acting on it. Since $\mathcal{J}^2 = \mathbf{1}$, there are $\mathbb{R}$-linear projectors $\frac{1}{2}(\mathbf{1} \pm \mathcal{J})$ splitting $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ into a direct sum of two $\mathbb{R}$-subspaces, one is fixed by $\mathcal{J}$, the other one is inverted by $\mathcal{J}$. For $\mathfrak{D} = \mathbb{R}$, $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ is 1-dimensional and the symmetry is fixed. The complexification of this case is obtained, if $\mathfrak{D} = \mathbb{C}$ and $\mathcal{J}$ is $\mathbb{C}$-linear, i.e., $\mathcal{J}$ does not involve complex conjugation. Otherwise both symmetries are possible. An algorithm for constructing a bilinear form with a specified symmetry has to generate a non-zero element W of $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$. If $W \pm W^{\mathcal{J}}$ is non-zero an appropriate element is found otherwise further $\mathbb{R}$-linearly independent elements of $\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ need to be generated. In any case fixing the symmetry of the bilinear form still leaves a real scale factor undetermined. Since for the quaternionic case, $\mathfrak{D} = \mathbb{H}$, one of the spaces $\frac{1}{2}(\mathbf{1} \pm \mathcal{J})(\mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}})$ has an $\mathbb{R}$-dimension bigger than 1, there are bilinear forms with the same symmetry factor $\theta = \pm 1$, but with different induced involutions j_W. These involutions are related by an automorphism of $\mathbb{H}$, which can be seen from the relationship of j_W and j_C:

$$\delta^{j_C} = \rho^{-1} \delta^{j_W} \rho. \tag{16}$$

(Note that $j_C = j_{W'}$ for $W' = \mathbb{P} \vee C^{-1} \vee \mathbb{P}^{\mathcal{J}} = \rho^{-1} W$.)

4. Example

Whereas the previous section is applicable to any simple algebra we will now turn to Clifford algebras for our example. In particular, we will demonstrate the usefulness of the alternative construction by classifying spin invariant $\mathbb{C}$-bilinear forms on Dirac spinors in even dimension $n = 2m$, where the complex Clifford algebra is indeed simple.

Since we assume $\mathbb{C}$-linearity in both arguments the signature of the metric is irrelevant and may be chosen Euclidean, i.e., we consider the Clifford algebra $\mathcal{C}l(n)$ over the complex vector space $V = \mathbb{C}^n$ with non-degenerate metric g. Let $\{e_i\}$ be an orthonormal basis of V, which implies that

$$e_i \vee e_j + e_j \vee e_i = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{17}$$

Then

$$\mathbb{P} = \frac{1}{2^m}(1 + ie_1 e_{m+1})(1 + ie_2 e_{m+2}) \cdots (1 + ie_m e_{2m}) \tag{18}$$

is a primitive idempotent. For the involutions $\mathcal{J} = \xi$, which just interchanges the order of products, and $\mathcal{J} = \xi\,\eta$, where η is the automorphism that inverts V we obtain

$$\mathbb{P}^{\xi} = \mathbb{P}^{\xi\,\eta} = \frac{1}{2^m}(1 - ie_1 e_{m+1})(1 - ie_2 e_{m+2}) \cdots (1 - ie_m e_{2m}) = 1 - \mathbb{P} \quad (19)$$

Therefore we have a non-zero element $W \in \mathbb{P} \vee \mathfrak{A} \vee \mathbb{P}^{\mathcal{J}}$ in both cases

$$W = \mathbb{P} \vee e_1 \vee \cdots \vee e_m \vee \mathbb{P}^{\mathcal{J}} = \frac{1}{2^m}(e_1 + ie_{m+1})(e_2 + ie_{m+2}) \cdots (e_m + ie_{2m}). \quad (20)$$

To obtain the symmetry of the induced bilinear forms we consider

$$W^{\mathcal{J}} = \mathbb{P} \vee e_m^{\mathcal{J}} \vee e_{m-1}^{\mathcal{J}} \vee \cdots \vee e_2^{\mathcal{J}} \vee e_1^{\mathcal{J}} \vee \mathbb{P}^{\mathcal{J}} = \begin{cases} (-1)^{\frac{m(m-1)}{2}} W & \text{for } \mathcal{J} = \xi, \\ (-1)^{\frac{m(m+1)}{2}} W & \text{for } \mathcal{J} = \xi\,\eta. \end{cases} \quad (21)$$

So for even m both involutions ξ and $\xi\,\eta$ are equivalent and they are inequivalent for odd m. The induced bilinear form is symmetric for $\mathcal{J} = \xi$ and $m \equiv 0, 1 \bmod 4$ or $\mathcal{J} = \xi\,\eta$ and $m \equiv 0, 3 \bmod 4$, otherwise it is antisymmetric.

5. The Principle of Inheritance

The ideas that give rise to the alternative approach to bilinear forms on spinors as described in section 3 can be traced to a deeper relationship between (anti-)automorphisms of a simple algebra and maps on its minimal ideals. We restrict ourselves to the presentation of the logical structure of the theory below, since a complete derivation is in close analogy to the development in section 3

Consider a general (anti-)automorphism $*$ ($\mathcal{J}$) of a simple algebra $\mathfrak{A}$. (Complex conjugation and one of the standard involutions, e.g., ξ, may serve as examples in the case of Clifford algebras.) Given such maps on $\mathfrak{A}$ we observe a first inheritance from the simple algebra to its minimal ideals. Namely, on any any minimal left idea $\mathfrak{A} \vee \mathbb{P}$ (or equivalently primitive idempotent $\mathbb{P}$) we have an induced map

$$\begin{aligned} \cdot^c : \mathfrak{A} \vee \mathbb{P} &\to \mathfrak{A} \vee \mathbb{P} \\ \psi &\mapsto \psi^c := \psi^* \vee B, \end{aligned} \quad (22)$$

for any non-zero element $B \in \mathbb{P}^* \vee \mathfrak{A} \vee \mathbb{P}$, and from a minimal left ideal to its dual

$$\begin{aligned} \tilde{\cdot} : \mathbb{P} \vee \mathfrak{A} &\to \mathfrak{A} \vee \mathbb{P} \\ \psi &\mapsto \tilde{\psi} := W \vee \psi^{\mathcal{J}}, \end{aligned} \quad (23)$$

for any non-zero element of $W \in I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$. Since both $I\!P^* \vee \mathfrak{A} \vee I\!P$ and $I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$ are 1-dimensional one-sided $\mathfrak{D}$-modules, namely a right and a left one, the maps (22) and (23) are defined up to elements of $\mathfrak{D}$. The crucial properties of these maps are:

$$(a \vee \psi)^c = a^* \vee \psi^c \quad \text{and} \quad (\widetilde{a \vee \psi}) = \tilde{\psi} \vee a^{\mathcal{J}}. \tag{24}$$

(Of course, as our notation is suggesting, these are defining properties for spinor charge conjugation and spinor adjoint, if $*$ is identified with complex conjugation and $\mathcal{J}$ is identified with one of the standard involutions.)

In a next step of inheritance we have induced maps on $\mathfrak{D} \equiv I\!P \vee \mathfrak{A} \vee I\!P$, namely an automorphism $\flat$

$$B\, \delta^\flat = \delta^* \vee B \tag{25}$$

and an antiautomorphism j

$$\delta^j\, W = W \vee \delta^{\mathcal{J}}, \tag{26}$$

which we have implicitly defined for $\delta \in \mathfrak{D}$. If $*$ and $\mathcal{J}$ are involutory then $\flat$ and j can be made involutory by choosing $B^* \vee B = \pm P$ and $W^{\mathcal{J}} = \pm W$, which is always possible for $\mathfrak{D} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$. Also, if $B^* \vee B = \pm P$ then $(\psi^c)^c = \pm\psi$.

Again, the crucial ingredient is the $\mathfrak{D}$-module structure of $I\!P^* \vee \mathfrak{A} \vee I\!P$ and $I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$, which relies on the fact that any two primitive idempotents are conjugates in a simple algebra. Of course, an algorithm for implementing charge conjugation can be found according to Eq. (22), where a non-zero $B \in I\!P^* \vee \mathfrak{A} \vee I\!P$ must be determined which can then be normalized to satisfy $B^* \vee B = \pm P$.

6. Summary

Based on the key observation that $I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$ is a left $\mathfrak{D}$-module (cp. Eqs. (1,2,10)), we have established that for given involution $\mathcal{J}$ and primitive idempotent $I\!P$ any non-zero $W \in I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$ determines a non-degenerate bilinear form according to Eq. (11). Its induced involution j_W and symmetry properties can be determined from Eqs. (12,13). A bilinear form of a specified symmetry can be obtained by choosing a non-zero element $W \in \frac{1}{2}(\mathbf{1} \pm \mathcal{J})(I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}})$.

Using this construction, a classification of spinor bilinear forms can be reduced to a classification of the behavior of the left $\mathfrak{D}$-module $I\!P \vee \mathfrak{A} \vee I\!P^{\mathcal{J}}$ under $\mathcal{J}$ for a given primitive idempotent $I\!P$ and involution $\mathcal{J}$. Since all primitive idempotents are related by an isomorphism only one primitive idempotent needs to be considered.

In computational terms an expression for a bilinear form of a specified symmetry can be found by generating linearly independent elements of $\mathbb{P}\,{}_\vee\mathfrak{A}{}_\vee\,\mathbb{P}^{\mathcal{J}}$ and taking the appropriate projection $\frac{1}{2}(\mathbf{1} \pm \mathcal{J})$. This algorithm avoids the introduction of a matrix basis which is necessary in order to solve for C^{-1}. Therefore the construction allows a purely symbolic yet efficient implementation of bilinear forms in computer algebra systems. For simplifications of purely symbolic expressions, we can even avoid an explicit form of W and merely use information about its properties such as j_W and θ (cp. Eqs. (12,13)).

We traced to constructive definition to a principle of inheritance of structures in the simple algebra to its minimal ideals. According to this principle a similar constructive approach to charge conjugation has been indicated using the structure of $\mathbb{P}^*\,{}_\vee\,\mathfrak{A}\,{}_\vee\,\mathbb{P}$ as a right $\mathfrak{D}$-module.

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