

CLIFFORD ALGEBRAIC REMARK ON THE MANDELBROT SET OF TWO-COMPONENT NUMBER SYSTEMS

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Abstract

We investigate with the help of Clifford algebraic methods the Mandelbrot set over arbitrary two-component number systems. The complex numbers are regarded as operator spinors in $\mathbf{D} \times \mathbf{spin}(2)$ resp. $\mathbf{spin}(2)$. The thereby induced (pseudo) normforms and traces are not the usual ones. A multi quadraic set is obtained in the hyperbolic case contrary to [1]. In the hyperbolic case a breakdown of this simple dynamics takes place.

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1. Introduction

The quadratic Mandelbrot set is the set of non divergent points in the iteration $z_{n+1} := z_n + c$, $z_0 := (0, 0), \forall c$, over the binary numbers, with the same norm as in the complex case. Recently a convergence proof for the quadratic Mandelbrot set was given [1]. This set was discussed numerically in [2]. The puzzling effect, that by changing only one sign in the iteration formula results in a completely different not even chaotic pictures, was expressed by the term “perplex numbers”, which are nothing but binary numbers [3, 4]. Where the disappearance of the chaotic behavior was expressed by “The mystery of the quadratic Mandelbrot set” in [1].

In literature we find a host of names for the algebraic structure over pairs $(1, \gamma)$ with $\gamma^2 = 1$. The complex numbers, earlier invented, came through Cardano, Euler, Argant, Gauss and others into physics. Of course the situation is not so clear in the other choice of sign. The names are ‘perplex’, binary, trivial, hyperbolic, unipodal [5] and there may be some others. We have chosen several names related to the topic under discussion. In the mathematical context we prefer with A. Hahn the notation trivial, because of the direct sum structure. In the geometric situation we prefer the geometric term hyperbolic, because of the underlying geometry. Unipodal, binary and perplex seems to have no direct meaning and are only used for reference issues.

In this note, we want to show how the two cases can be understood in a Clifford algebraic framework. This provides us with several advantages over the usual picture.

First we introduce the complex numbers as operator spinors acting on a unit reference vector in the Euclidean vector space $\mathbf{E}(2)$. This is in analogy with Kustaanheimo, Stiefel [6], Lounesto [7] and the inspiring chapter 8 “Spinor Mechanics” of Hestenes’ “New Foundation of classical Mechanics” [8].

In a second step we utilize Clifford algebras with arbitrary not necessarily symmetric or antisymmetric bilinear forms. Such algebras were discussed by Chevalley [9], Riesz [10], Oziewicz [11] and Lounesto [12] in a mathematical and geometrical manner.

This procedure is a general tool and was used in [13] in an entire other context. Here the geometrical point of view allows a generalization of the concept of operator spinors to those vector spaces, which are equipped with general non degenerate bilinear forms.

The third step stems from the observation, that the Clifford algebra is strongly connected with a quadratic algebra. This structure provides us with several existence and uniqueness theorems. Separability provides us with a unique (pseudo) normform, as with a unique trace [14].

Surprisingly we are left with a norm different from the one used in [1]. But here the (pseudo) normform is compatible with the algebraic structure and the geometrical meaning. These (pseudo) normforms and traces are therefore preferable.

Using these norms, we clearly see other images in numerical experiments. The obtained pictures are in agreement with the physical situation at hand. Several special cases will be discussed. The Mandelbrot dynamic is shown to be incompatible with this structure and a better one should be derived from inhomogeneous $\mathbf{D} \times \mathbf{Ispin}$ or \mathbf{Ispin} groups. The breakdown of the dynamics in the hyperbolic case may be interpreted as a decay process or an absorption of the particle considered.

In section two we introduce the operator spinor for $\mathbf{E}(2)$ ¹. In section three we introduce Clifford algebras with nonsymmetric bilinear forms, described explicitly in [11]. In section four we recall several facts from the theory of quadratic algebras as proposed in [14]. In section five we discuss the Mandelbrot set over arbitrary two-component number systems using the (pseudo) normforms induced by algebraic and geometric considerations. The conclusion

¹ See Hestenes [8] for an enlargement to 3-dimensional space and the spinor gauge formulation of the Kepler motion.

summarizes our results and compares it to other work done.

2. Complex Numbers as Operator Spinors on $\mathbf{E}(2)$.

In this section we introduce the concept of algebraic spinors in the sense of Hestenes [8, 15] and others [9, 10]. Therein a geometrical meaning is given to the complex numbers. Indeed they are commonly identified with coordinates, which is quite obscure. There is a long quest in mathematics to give a geometrical meaning to the complex entities. In spite of their natural occurring in algebraic geometry they remained somehow mysterious [16]. An attempt in this direction seems very useful in the light of the ubiquitous appearance of complex numbers and *complex coordinates* in physics.

We choose a standard orthonormal basis $e_i, i \in (1, 2)$ of $CL(\mathbf{E}(2), \delta)$ with the properties (algebra product by juxtaposition)

$$\begin{aligned} e_i e_i &= e_i^2 = 1 && \text{normalized} \\ e_i e_j + e_j e_i &= 0, \quad i \neq j; && \text{orthogonality.} \end{aligned} \tag{1}$$

The standard involution (conjugation) may be defined by

$$\begin{aligned} J &: CL \rightarrow CL \\ J &: J(ab) := J(a)J(b) \\ J &: J|_{\mathbf{K} \oplus \mathbf{E}(2)} := id_{\mathbf{K}} - id_{\mathbf{E}(2)} \\ J^2 &= id_{CL}. \end{aligned} \tag{2}$$

The reversion is the main antiautomorphism defined by (see [17])

$$\begin{aligned} \sim &: CL \rightarrow CL \\ \sim &: (ab)^\sim := b^\sim a^\sim \\ \sim &: \sim|_{\mathbf{K} \oplus \mathbf{E}(2)} := id_{\mathbf{K} \oplus \mathbf{E}(2)} \\ (\sim)^\sim &= id_{CL}, \end{aligned} \tag{3}$$

where \mathbf{K} and $\mathbf{E}(2)$ are the images of the field (ring) and the vector space (module) in CL . Because there is a natural injection we will not distinguish between these pictures.

An algebra basis is given by

$$\{X_I\} := \{1, e_2 \wedge e_1 e_1, e_2\} \tag{4}$$

with

$$\begin{aligned}
e_i \wedge e_j &:= \frac{1}{2}(e_i e_j + J(e_j) e_i) \\
e_i \lrcorner e_j &:= \langle e_i | e_j \rangle := \delta_{ij} = \frac{1}{2}(e_1 e_2 - J(e_2) e_1).
\end{aligned} \tag{5}$$

We may use the symbol ‘ i ’ to denote $e_2 \wedge e_1$, because of the properties

$$\begin{aligned}
i^2 &= (e_2 \wedge e_1)(e_2 \wedge e_1) = e_2 e_1 e_2 e_1 = -e_2^2 e_1^2 = -1 \\
[i, X_I]_{deg|X_I} &= 0.
\end{aligned} \tag{6}$$

Where $deg|X_I$ means the \mathbf{Z}_2 -grade of the homogeneous element X_I . For (odd) even elements we have the (anti)commutator. This graduation is induced by J and has the structure

$$CL = CL_+ \oplus CL_-, \tag{7}$$

where the even (+) part constitutes a subalgebra, whereas the odd (-) part has only a CL_+ module structure. In the $\mathbf{E}(2)$ case, CL_+ is generated by $\{1, i\}$ and is itself a Clifford algebra isomorphic to the field \mathbf{C} of complex numbers.

Now we want to emphasize the operator character of complex numbers. Therefore we calculate the left action of the element ‘ i ’ on the base vectors

$$\begin{aligned}
ie_1 &= e_2 \\
ie_2 &= -e_1,
\end{aligned} \tag{8}$$

which is a counter clockwise rotation by $\pi/2$. If we choose an arbitrary unit reference vector e , $e^2 = 1$, we may write the elements in CL_+ as

$$CL_+ \ni z := x + yi, \quad x, y \in \mathbf{K}. \tag{9}$$

So we have

$$z = ze^2 = (xe + yie)e = ve \tag{10}$$

with $v \in \mathbf{E}(2)$. From

$$\{e, ie\}_+ = eie + iee = -ie^2 + ie^2 = 0 \tag{11}$$

we have $e \perp ie$. Thus $\{e, ie\}$ span $\mathbf{E}(2)$, and we are able to rotate the coordinate system by an orthogonal transformation to achieve

$$\begin{aligned} e &= e_1 \\ ie &= e_2. \end{aligned} \tag{12}$$

The map $z \rightarrow ze$ is a bijective map from $CL_+ \rightarrow \mathbf{E}(2)$, because e is invertible. On the other hand we may look on the map $z : \mathbf{E}(2) \rightarrow \mathbf{E}(2)$ with $za \rightarrow v$ for arbitrary $a \in \mathbf{E}(2)$ and fixed $z \in CL_+$.

Only with the choice $e = e_1$ we are able to interpret the scalar and bivector part of an operator spinor to be coordinates.

The modulus may be defined as follows

$$\begin{aligned} |z|^2 &:= zz^\sim = (x + iy)(x - iy) = x^2 + y^2 \geq 0 \\ |z|^2 = 0 &\Leftrightarrow z \equiv 0 \end{aligned} \tag{13}$$

CL_+ is isomorphic to the field \mathbf{C} and therefore algebraically closed. Thus it is possible to find a root

$$z = w^2 = zz^\sim \frac{z}{zz^\sim} = |z|u^2 = (|z|^{\frac{1}{2}}u)^2, \quad uu^\sim = 1, \quad |z| \in \mathbf{K} \sim \mathbf{D}. \tag{14}$$

\mathbf{D} describes then a dilatation and ω is unique up to a sign. We can reformulate the map as

$$ze = w^2e = |z|^2u^2e = |z|^2ueu^\sim = (|z|u)e(|z|u^\sim). \tag{15}$$

This decomposes the left action of z into a dilatation and a spinorial rotation. Because of $u^\sim = u^{-1}$, u is a **spin (2)** transformation. The transformation obtained by the left action of u is of half angle type.

This point of view is independent of the special vector e and emphasizes as well the operator character of the iteration formula. Indeed the iteration is a sequence of maps from $\mathbf{E}(2) \rightarrow \mathbf{E}(2)$, where $a \in \mathbf{E}(2)$ is given by $z_n(e)$.

This works well, because $CL_+ \cong \mathbf{C}$ is algebraically closed and products as well as sums of CL_+ elements yields new CL_+ elements, which can be interpreted as new operator spinors z' . In the general higher dimensional case, as in the hyperbolic one, only the multiplicative structure forms a (Lipschitz) group $\mathbf{D} \times \mathbf{spin}(\mathbf{p}, \mathbf{q})$, whereas the additive group is in general incompatible with the geometric structure. For example we find two timelike vectors in the forward light cone, which become space like when added.

A physically sound dynamical model should therefore have an invariant, i.e. multiplicative structure. By studying multiplicative structures in vector spaces admitting one higher dimension and performing a split [15, 18]

one can achieve affine transformations with $\mathbf{D} \times \mathbf{Ispin}(\mathbf{p}, \mathbf{q})$. For clarity and brevity, as well as for comparison with the results of the literature we omit this complication. However we take it into account, when we perform the actual calculations.

3. Clifford Algebras for Arbitrary Bilinear Forms.

In this section we give a brief account on Clifford algebras with arbitrary bilinear forms. It's main purpose is to show the connection to quadratic algebras. For a somehow polemic discussion of Grassmann or Clifford algebra as a basic tool in physics we refer to [11, 19].

At first glance it is surprising to have non symmetric bilinear forms in Clifford algebras, because in the usual approach [20] they arise naturally with symmetric bilinear or sesquilinear forms. The situation looks even more puzzling when noticing the universality of Clifford algebras, usually stated as: "There is up to isomorphisms only one unique algebraic structure compatible with a quadratic form of signature (p, q) on the space V ".

Why is it worse to study isomorphic structures? In [13] it was demonstrated that the physical content of a theory depends sensible on the embedding $\wedge \mathcal{V} \rightarrow CL(\mathcal{V}, B)$ of the Grassmann or exterior algebra into the Clifford algebra. For example normal ordering is exactly such a change of this embedding.

Here we will give the connection between such changes and the properties of quadratic algebras. They provide us with a deeper geometric understanding as described in section four.

Chevalley [9] was the first who constructed the Clifford algebra product from two parts. These parts, then explicitly, exhibit the twofold structure of the Clifford elements. As an endomorphism, Clifford elements are in the universal enveloping algebra built over $\mathcal{V} \oplus \mathcal{V}^*$.

One part acts like a derivation on the space $\wedge \mathcal{V}$ of exterior powers of \mathcal{V} . Especially if $\omega_x = x \rfloor$ is a form of degree 1 it constitutes a form on \mathcal{V} into the field \mathbf{K} ($x, y \in \mathcal{V}$, $\rfloor : \mathcal{V} \rightarrow \mathcal{V}^*$),

$$\omega_x(y) = x \rfloor y := B(x, y) \in \mathbf{K}. \quad (16)$$

Thus \rfloor is a dual isomorphism parametrized by B . The second part of the Clifford product is simply the exterior multiplication.

Marcel Riesz [10] reexpressed the contraction and the exterior multiplication with a grade involution J and the Clifford product. He obtained for $\text{char } \mathbf{K} \neq 2$, $x \in \mathcal{V}$, $u \in CL$

$$\begin{aligned}
 x \rfloor u &:= \frac{1}{2}(xu - J(u)x) \\
 x \wedge u &:= \frac{1}{2}(xu + J(u)x).
 \end{aligned}
 \tag{17}$$

One is able to extend these operations, over a suitable field (ring, module), to higher degrees of multivectors by $(a, b, c \in \mathcal{V}, X \in CL)$

$$\begin{aligned}
 (a \wedge b) \rfloor X &= a \rfloor (b \rfloor X) \\
 a \rfloor (bc) &= (a \rfloor b)c + J(b)(a \rfloor c).
 \end{aligned}
 \tag{18}$$

Associativity of the wedge product and other properties are shown in [10]. A detailed study of Clifford algebras with non symmetric bilinear forms was given in [11, 9].

In the following it is convenient to use explicit not necessarily normed or orthogonal generating sets. Especially for low dimensional examples this will be useful. Therefore we define the left-contraction and the exterior product as $(\mathcal{V} = \text{span}\{e_1, \dots, e_n\})$

$$\begin{aligned}
 e_i \rfloor e_j &:= B(e_i, e_j) \cong [B_{ij}] \\
 e_i \wedge e_j &:= \frac{1}{2}(e_i e_j - e_j e_i)
 \end{aligned}
 \tag{19}$$

with the standard grade involution $J(e_i) = -e_i$.

Next we may decompose B into symmetric and antisymmetric parts

$$\begin{aligned}
 G + F &:= B \\
 G^T &= G \\
 F^T &= -F
 \end{aligned}
 \tag{20}$$

where T is the matrix transposition. In the case of an algebra over the complex numbers one has to use Hermitean adjoint, but here we deal exclusively with real algebras.

We specialize now to 2 dimensions. Then we are left with 4 parameters, 3 of the symmetric and 1 of the antisymmetric part.

$$[G] = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} \quad (21)$$

$$[F] = \begin{pmatrix} 0 & F_{12} \\ -F_{12} & 0 \end{pmatrix}.$$

The even part of this algebra satisfies a quadratic equation for every element $z = x + ye_2 \wedge e_1$ where

$$\begin{aligned} (e_2 \wedge e_1)^2 &= (e_2 e_1 - B_{21})(e_2 \wedge e_1) \\ &= e_2(e_1](e_2 \wedge e_1)) - B_{21}e_2 \wedge e_1 \\ &= e_2(B_{12}e_1 - B_{11}e_2) - B_{21}e_2 \wedge e_1 \\ &= (B_{12} - B_{21})e_2 \wedge e_1 - (B_{11}B_{22} + B_{12}B_{21}) \\ &= -2F_{21}e_2 \wedge e_1 - \det(B). \end{aligned} \quad (22)$$

Now we expand $\det(B)$

$$\begin{aligned} \det(B) &= B_{11}B_{22} - B_{12}B_{21} \\ &= G_{11}G_{22} - (G_{12} + F_{12})(G_{12} - F_{12}) \\ &= F_{12}^2 + \det(G) \end{aligned} \quad (23)$$

and we arrive at

$$(e_2 \wedge e_1)^2 = 2F_{12}e_2 \wedge e_1 - \det(G) - F_{12}^2. \quad (24)$$

If there is no antisymmetric part (e.g. $F_{12} \equiv 0$), we specialize to

$$(e_2 \wedge e_1)^2 = -\det(G) \quad (25)$$

and the determinant of G describes the geometry at hand. If $\det(G)$ is positive we arrive at an (anti) Euclidean geometry. The sign of the square of a base vector e_i , is not yet specialized, but both of them are required to have the same sign. In the negative case the geometry is hyperbolic.

We are free to choose another algebra basis via a new dotted wedge product defined by

$$e_i \dot{\wedge} e_j := F_{ij} + e_i \wedge e_j. \quad (26)$$

Therefore we have incorporated the whole antisymmetric part in the wedge product and the Clifford product decomposes as

$$\begin{aligned} e_i e_j &= B_{ij} + e_i \wedge e_j \\ &= G_{ij} + e_i \dot{\wedge} e_j. \end{aligned} \tag{27}$$

We define a new set of elements spanning the algebra, which reads in two dimensions

$$\{Y_J\} := \{1, e_2 \dot{\wedge} e_1, e_1, e_2\}. \tag{28}$$

This leads to another quadratic relation

$$\begin{aligned} (e_2 \dot{\wedge} e_1)^2 &= (e_2 \wedge e_1 + F_{21})^2 \\ &= (e_2 \wedge e_1)^2 + 2F_{21} e_2 \wedge e_1 + F_{21}^2 \\ &= 2(F_{21} + F_{12}) e_2 \wedge e_1 - \det(G) - F_{12}^2 + F_{21}^2 \\ &= -\det(G). \end{aligned} \tag{29}$$

In this construction the linear term has been absorbed in the dotted wedge. The quadratic relation has simplified to the homogeneous case discussed above, but now not as a special case.

Next we introduce a (pseudo) norm function on the quadratic subalgebra. In the Clifford algebra there are several such constructions, known as spinor norms [21, 22].

For vector elements $a \in \mathcal{V}$ every of the three following maps has the image in the field \mathbf{K} . With the standard involution $J(a) = -a$ we have

$$a^2 \rightarrow \mathbf{K}; \quad aJ(a^\sim) = aJ(a)^\sim = -a^2 \rightarrow \mathbf{K}; \quad aa^\sim = a^2 \rightarrow \mathbf{K} \tag{30}$$

But applying this to bivector elements, the first relation maps not in the field, but is exactly the quadratic relation derived above. If we allow arbitrary involutions, a and $J(a)$ need not be parallel vectors, as is the same for bivector elements. In general we may not expect this map to be scalar valued. To analyze the third map, we recognize

$$\begin{aligned} (e_i \wedge e_j)^\sim &= (e_i e_j - B_{ij})^\sim = e_j e_i - B_{ij} \\ &= e_j \wedge e_i + B_{ji} - B_{ij} \\ &= e_j \wedge e_i - 2F_{ij} \\ &= -e_i \wedge e_j - 2F_{ij}. \end{aligned} \tag{31}$$

We may define the adjunct of a matrix by performing the determinant over a restricted index set. Thus we have

$$Adj_{ij}(B) := B_{ii}B_{jj} - B_{ij}B_{ji}, \quad (32)$$

which is the two-by-two determinant relative to the indices i, j . Hence we have with (22) (no sums involved)

$$\begin{aligned} (e_i \wedge e_j)(e_i \wedge e_j)^\sim &= -(e_i \wedge e_j)^2 - 2F_{ij}e_i \wedge e_j \\ &= Adj_{ij}(B) + 2F_{ij}e_i \wedge e_j - 2F_{ij}e_i \wedge e_j \\ &= Adj_{ij}(B) \end{aligned} \quad (33)$$

as with (29) in the same manner for the dotted case

$$\begin{aligned} (e_i \dot{\wedge} e_j)(e_i \dot{\wedge} e_j)^\sim &= -(e_i \dot{\wedge} e_j)(e_i \dot{\wedge} e_j) \\ &= Adj_{ij}(G). \end{aligned} \quad (34)$$

This reduces in the case of two dimensions to

$$(e_2 \dot{\wedge} e_1)(e_2 \dot{\wedge} e_1)^\sim = Adj_{21}(G) = Adj_{12}(G) = \det(G), \quad (35)$$

which is, up to a factor -4, the discriminant of the quadratic equation (29). If the discriminant is positive ($\det(G) < 0$), there are two roots in the algebra (over \mathbf{R}), whereas otherwise the field has to be algebraically closed (i.e. \mathbf{C}), or extracting the root is not possible.

We introduce the abbreviation $X \cong \{e_2 \wedge e_1 \text{ or } e_2 \dot{\wedge} e_1\}$, $a \cong \{2F_{12} \text{ or } 0\}$, $b \cong \{-\det(B) \text{ or } -\det(G)\}$ and may write the quadratic algebra as

$$\frac{K[X]}{X^2 - aX - b'} \quad (36)$$

the polynom algebra generated by X over the field \mathbf{K} modulo the quadratic relation. This form is important for comparison with the theory of quadratic algebras, but our aim is the exposition of the connection to the geometric relations encoded in this formula.

4. Quadratic Algebras, Conjugation and Special Elements.

In this section we give some results exposed in [14], which provide us with existence and uniqueness theorems. This supplies more fondness to our somehow loosely construction above.

The constructions are valid in much more general settings, as Clifford algebras over finite fields, rings or over modules, which is yet not needed but quite interesting.

Let \mathcal{R} be a commutative ring and \mathcal{I} an ideal of \mathcal{R} , then $\mathcal{R} \rightarrow \mathcal{R}[X]/\mathcal{I} = \mathcal{A}$ is in a natural way a \mathcal{R} -algebra.

A free quadratic algebra is obtained if one factorizes the polynomial algebra in the indeterminate X over \mathcal{R} by the ideal $\mathcal{I} = (X^2 - aX - b)$

$$S = \frac{\mathcal{R}[X]}{X^2 - aX - b}. \tag{37}$$

The identity is

$$1_S = 1 + \mathcal{I} = 1 + (X^2 - aX - b) \tag{38}$$

and because $r \in \mathcal{R}$, $r \rightarrow r1_S$ is injective we have $\mathcal{R} \subseteq S$. A basis of S as \mathcal{R} -module is given by

$$\{1, v = X + \mathcal{I} = X + (X^2 - aX - b)\}. \tag{39}$$

It follows that

$$v^2 = av + b. \tag{40}$$

We have several isomorphisms:

$$S_1 = \frac{\mathcal{R}[X]}{X^2 - X} \cong \mathcal{R} \oplus \mathcal{R} \tag{41}$$

with the diagonal product map. $\phi : \mathcal{R} \oplus \mathcal{R} \rightarrow S_1$ is an isomorphism and we denote S_1 as trivial quadratic \mathcal{R} -algebra. This is the ‘perplex’ case from above!

$$S_2 = \frac{\mathcal{R}[X]}{X^2} \tag{42}$$

constitutes the algebra of dual numbers.

Denoting the units of \mathcal{R} as \mathcal{R}^* , we can formulate the isomorphism criterion [14](1.1).

Let $\mathcal{R}[X]/(X^2 - aX - b)$ and $\mathcal{R}[X]/(X^2 - cX - d)$ be quadratic algebras over \mathcal{R} . Then

$$\frac{\mathcal{R}[X]}{X^2 - aX - b} \cong \frac{\mathcal{R}[X]}{X^2 - cX - d}$$

iff there exist elements $r \in \mathcal{R}$ and $u \in \mathcal{R}^*$ such that

$$\begin{aligned} (i) \quad c &= ua + 2r \\ (ii) \quad d &= u^2b - rua - r^2. \end{aligned} \tag{43}$$

Now, has $X^2 - aX - b$ a root, say γ in \mathcal{R} then we have

$$X^2 - aX - b = (X - \gamma)(X - (a - \gamma)) \tag{44}$$

and $a - \gamma$ is another root (in \mathcal{R}) of the quadratic equation. If $a - \gamma = \gamma$ then γ is a double root. We can state the following

- (1) $\frac{\mathcal{R}[X]}{X^2 - aX - b} \cong \frac{\mathcal{R}[X]}{X^2 - taX - t^2b}$ for $t \in \mathcal{R}^*$
- (2) $\frac{\mathcal{R}[X]}{X^2 - aX - b} \cong \frac{\mathcal{R}[X]}{X^2 - cX} \Leftrightarrow X^2 - aX - b$ has a root in \mathcal{R}
- (3) $\frac{\mathcal{R}[X]}{X^2 - aX - b} \cong \frac{\mathcal{R}[X]}{X^2} \Leftrightarrow X^2 - aX - b$ has a double root in \mathcal{R}

Examples:

1. $\mathcal{R} \cong \mathbf{C}$: The only quadratic algebras are $\mathbf{C}[X]/(X^2 - X)$ the complex trivial algebra and $\mathbf{C}[X]/(X^2)$ the algebra of complex dual numbers, because of the existence of roots for every $X \in \mathbf{C}$.
2. $\mathcal{R} \cong \mathbf{R}$: We have three cases, because negative numbers posses no roots in \mathbf{R} .

$$S \cong \begin{cases} \frac{\mathbf{R}[X]}{X^2 - X} & a^2 + 4b > 0 \quad \text{trivial, 'perplex' } \\ \frac{\mathbf{R}[X]}{X^2} & a^2 + 4b = 0 \quad \text{dual numbers} \\ \frac{\mathbf{R}[X]}{X^2 + 1} \cong \mathbf{C} & a^2 + 4b < 0 \quad \text{complex numbers} \end{cases} \tag{45}$$

Of course, $a^2 + 4b$ is the discriminant of the quadratic relation.

3. $\mathcal{R} \cong \mathbf{Z}$: Results in the infinitely many isomorphism classes

$$\frac{\mathbf{Z}[X]}{X^2 - aX - b} \cong \begin{cases} \frac{\mathbf{Z}[X]}{X^2 - n} & \text{if } a \text{ is even} \\ \frac{\mathbf{Z}[X]}{X^2 - X - n} & \text{if } a \text{ is odd} \end{cases} \tag{46}$$

It turns out, that simpler rings (as also finite Galois fields) bear much more structure.

If α is an (anti) automorphism and $\alpha^2 = id_{\mathcal{A}}$, then α is an involution. Algebra homomorphisms which preserve such a structure are called graded homomorphism.

In physics the super symmetric transformations (mixing of Grassmann parity) are not grade preserving and thus minor symmetric.

In a quadratic algebra we may introduce the involution $\sigma : S \rightarrow S$ on the base

$$\begin{aligned} \{1, v = X + (X^2 - aX - b)\} \\ 1^\sigma = 1 \\ v^\sigma = (a - v) \end{aligned} \tag{47}$$

which results in

$$\begin{aligned} (x + yv)^\sigma &= (x + ya) - yv \\ (x + yv)^{\sigma\sigma} &= (x + yv). \end{aligned} \tag{48}$$

This involution interchanges the roots of the quadratic relation (44). In the complex case this is the ordinary complex conjugation.

Now this kind of involution is ‘standard’ in quadratic algebras and induces the \mathbf{Z}_2 -grading of the Clifford algebra. Let $\mathcal{A}_i, i \in (0, 1)$ be \mathcal{R} -submodules and $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. As $\mathcal{R}1 \subseteq \mathcal{A}_0, \mathcal{A}$ is a \mathbf{Z}_2 -graded algebra. Of course the grading may be trivial, $\mathcal{A}_1 \equiv \{\}$. Elements $a \in \mathcal{A}_i$ ($\partial a = i$, grade of a) are called homogeneous.

We have two possibilities to introduce tensor products in graded algebras via

$$\begin{aligned} \mathcal{A} \otimes_{\mathcal{R}} B : (a \otimes b)(a' \otimes b') &:= (aa' \otimes bb') \\ \mathcal{A} \hat{\otimes}_{\mathcal{R}} B : (a \otimes b)(a' \otimes b') &:= (-)^{\partial a' \partial b} (aa' \otimes bb'). \end{aligned} \tag{49}$$

To the algebra \mathcal{A} we find the opposite algebra \mathcal{A}^{op} by reversing the product, $(ab)^{op} = b^{op}a^{op}$, which is a map from \mathcal{A} onto \mathcal{A}^{op} . Hence we construct the enveloping algebra $\mathcal{A}^e := \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^{op}$, as a $(\mathcal{A}, \mathcal{A})$ -bimodule. There is a unique homomorphism $\Phi : \mathcal{A}^e \rightarrow \mathcal{A}$, which satisfies $\Phi(a \otimes b^{op}) = ab$. If there exists also an homomorphism $\Theta : \mathcal{A} \rightarrow \mathcal{A}^e$ (coproduct) such that $\Phi\Theta = id_{\mathcal{A}}$ then the algebra is called separable. It follows then $\mathcal{A}^e = \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^{op} = ker(\Phi) \oplus \Theta(\mathcal{A})$. With [14](2.1)(2.3) we state:

1. \mathcal{A} is a separable \mathcal{R} -algebra iff \mathcal{A} has a separability idempotent, $e = \Theta(1)$.
2. The separability idempotent of a free quadratic algebra S over \mathcal{R} is unique.

Now it is possible to classify separable free quadratic \mathcal{R} -algebras by introducing the group $Qu_f(\mathcal{R})$. Therefore define

$$Q := \{(a, b) | a^2 + 4b \in \mathcal{R}^*\} \quad (50)$$

with the product

$$(a, b) * (c, d) := (ac, a^2d + c^2b + 4bd) \quad (51)$$

and the quotients $[a, b] = (a, b)/\mathcal{R}^{*2}$ form the group

$$Qu_f(\mathcal{R}) := \{[a, b] | (a, b) \in Q\}. \quad (52)$$

The cardinality of $Qu_f(\mathcal{R})$ is the number of isomorphism classes of (nontrivial) quadratic algebras, e.g.

$$\begin{aligned} Qu_f(\mathbf{C}) &\cong \frac{\mathbf{C}^*}{\mathbf{C}^{*2}} = 1 \\ Qu_f(\mathbf{R}) &\cong \frac{\mathbf{R}^*}{\mathbf{R}^{*2}} = \mathbf{Z}_2 \end{aligned} \quad (53)$$

This can be extended to a graded version $QU_f(\mathcal{R})$.

One observes a connection between grading and standard involution via the special elements. To see this, let $M = (\mathcal{V}, B)$ be the pair of a vector space with a bilinear form (or a quadratic module), then we can build the Clifford algebra $CL(M) = CL(\mathcal{V}, B)$. We get now [14](5.4)

The decomposition $CL(M) = CL_0(M) \oplus CL_1(M)$ is a (\mathbf{Z}_2) grading of $CL(M)$. $CL_0(M)$ is a subalgebra and $CL_1(M)$ is a $CL_0(M)$ module.

We call σ a standard involution, if σ is an antiinvolution and $aa^\sigma \in \mathcal{R} \forall a \in CL(M)$. In this case we define a (pseudo) norm and a trace as

$$\begin{aligned} nr(a) &:= aa^\sigma (\in \mathcal{R}) \\ tr(a) &:= a + a^\sigma (\in \mathcal{R}). \end{aligned} \quad (54)$$

An element $z \in CL(M)$ is a special element if $\{1, z\}$ is a basis of the centralizer $Cen_{CL(M)} CL_0(M) = \{c \in CL(M) | cd = dc \forall d \in CL_0(M)\}$. One can conclude that²

1. if rank M is odd, then $z \in CL_1(M)$, $z^\sigma = -z$ and $z^2 = b$ with $b \in \mathcal{R}$.
2. if rank M is even, then $z \in CL_0(M)$, $z^\sigma = a - z$ and $z^2 = az - b$ with $a, b \in \mathcal{R}, a^2 + 4b \in \mathcal{R}^*$.

² See [14] (8.3)

If γ is a root of $X^2 - aX - b$ in S , then S has the grading $S = S_0 \otimes S_1$, where

$$\begin{aligned} S_0 &= \text{Cen}_S(\gamma) = \{s \in S \mid \gamma s = s\gamma\} \\ S_1 &= \{s \in S \mid \gamma s + s\gamma = as\}. \end{aligned} \tag{55}$$

The grading is trivial if γ is in the center of S .

These properties are strongly interwoven and can be used in constructing representations of Clifford algebras [23].

The existence of special elements in a general setting is proved in chapter 10 of [14]. All this constructions are possible in higher dimensions, but our above naive geometric interpretation has then to be refined.

As a last topic we have a look at the representations of Clifford algebras. An homomorphism Φ of \mathcal{R} -algebras

$$\Phi : CL(M) \rightarrow \text{End}_S(P) \tag{56}$$

where M is a quadratic module, S a \mathcal{R} -algebra and P a right S -module is called an (S -)representation of $CL(M)$. In [14](8.7)(8.8) the connection between division algebras, gradings and the quadratic algebra is shown and connected with the existence of roots in S . The quadratic algebra S appears as tensor factor in such representations.

Despite the universality of Clifford algebras, we need for physical applications norms and traces, as the grading (vector space and \mathbf{Z}_2). Therefore we have to distinguish between such homomorphisms preserving these additional structures and those doing this not. The free quadratic groups etc. characterize the isomorphism classes available, where the kernel of the factorization parameterizes distinct but isomorphic representatives. In fact we fix a representant out of a whole class of isomorphic algebras by the simple choice of an antisymmetric part F . This concept is thus beyond a pure Clifford algebraic consideration.

One could try to enlarge the concept of a Clifford algebra in another way, that is by using a more involved sort of ideal. This was done in [24]. The factorization of the new algebra out of the tensor algebra is a difficult task. One is able to construct in such a way the Clifford algebra of a Hecke-braid, if some conditions hold. Despite this quite different approaches, the methods are intimately related, as can be shown in the case $q^n = 1$, where one obtains finite representations in the Hecke case also. This sort of relation will be established elsewhere.

5. The 2-dim. Case and Numeric Experiments.

In this section we discuss the 2-dim. case over \mathbf{R} . Because of the even dimension we expect to have a linear term in the quadratic relation and therefore a rich structure. Using the natural given standard involution, we can construct (pseudo) norms and traces as above and investigate the Mandelbrot set in each of the three cases. We do not get a quadratic set, but a ‘light cone’ structure in the hyperbolic case.

Let us choose a generating set $\{e_1, e_2\}$ as in (4) or in (28), and a bilinear form $B = G + F$ as above. We denote the bivector element as γ . Here one has to take care, because a change in the quadratic relation of γ is related with a change of the representation of the algebra. In the same time this results in a redefining of the wedge, as above done by using the two extreme cases \wedge and $\dot{\wedge}$.

We interpret the iteration formula in the operational sense explained in section two and plot all figures in the $\{1, \gamma\}$ -plane. A translation into \mathcal{V} may be done as explained also in section two. For comparison to other results this is here *not* done.

With Sylvester’s theorem we could achieve a diagonal form for the symmetric part of G with $G \equiv \text{diag}\{\pm 1, \pm 1\}$. But a rescaling of the base vectors by $\sqrt{|G_{11}|}$ and $\sqrt{|G_{22}|}$ would affect the magnitude of the nonsymmetric part also. This is contained in the isomorphism criterion and will therefore be done only in the quadratic algebra and not in the whole Clifford algebra.

Define $S = \mathcal{R}[\gamma]/(\gamma^2 - a\gamma - b)$. We get from (23)

$$\begin{aligned}\gamma^2 &= 2F_{12}\gamma - \det(B) \\ &= 2F_{12}\gamma - \det(G) - F_{12}^2.\end{aligned}\tag{57}$$

Hence we set

$$\begin{aligned}a &:= 2F_{12} \\ b &:= -\det(B) = -\det(G) - F_{12}^2.\end{aligned}\tag{58}$$

The discriminant d is connected to the metric via

$$d = a^2 + 4b = -4\det(G).\tag{59}$$

The discriminant of the quadratic relation is thus -4 times the determinant of the symmetric part of the bilinear form!

If $\det(G) = 0$ we have a double root in the algebra S . If the algebra is not dual (double root) we have $\det(G) \neq 0$.

As explained in section four, there exists a standard involution if the algebra S is separable. In our case this is the reversion in CL . It is constructed in the base $\{1, \gamma\}$ by

$$\begin{aligned} 1^\sigma &= 1 \\ \gamma^\sigma &= (a - \gamma) = 2F_{12} - \gamma. \end{aligned} \tag{60}$$

The inhomogeneous additive term is quite uncommon in usual approaches to Clifford algebras. Let us emphasize, that in the quantum mechanical case, where the field is complex, we are able to find always an algebra isomorphism to achieve $\gamma'^\sigma = \gamma'$. Because there is only one nontrivial isomorphism class. This may be an argument for using complex numbers in quantum mechanics.

The iteration formula is now obtained as follows

$$z := z_x + z_y \gamma; \quad c := c_x + c_y \gamma \tag{61}$$

and with

$$z_{n+1} := z_n^2 + c \tag{62}$$

we obtain

$$\begin{aligned} z_{x(n+1)} &= z_{x(n)}^2 + bz_{y(n)}^2 + c_x \\ z_{y(n+1)} &= 2z_{x(n)}z_{y(n)} + az_{y(n)}^2 + c_y. \end{aligned} \tag{63}$$

The spinor (pseudo) norm is then given by

$$nr^2(z) = zz^\sim = z_x^2 - bz_y^2 + az_xz_y. \tag{64}$$

We have:

- i) $nr^2(z)$ is positive definite in the complex isomorphism class
- ii) $nr^2(z)$ is positive semidefinite in the dual isomorphism class
- iii) $nr^2(z)$ is indefinite in the trivial isomorphism class (hyperbolic case)

Proof: We distinguish two cases

- a) Suppose that $y \equiv 0$. We are left with $nr^2(z) = x^2$, which is positive definite in x and nr^2 is semidefinite in case iii), by isomorphy to the case $a = b = 0$.
- b) Suppose $y \neq 0$. We introduce $\rho = x/y$ and have

$$\begin{aligned}
\frac{nr^2}{y^2} &= \rho^2 - ap - b \\
&= \left(\rho - \frac{a}{2}\right)^2 - \left(\frac{a^2}{4} + b\right) \\
&\geq -\left(\frac{a^2}{4} + b\right) = -d = 4\det(G).
\end{aligned} \tag{65}$$

Thus if the discriminant is negative the norm is positive definite. If $d = 0$, the norm may be zero for non-null elements, that is positive semidefinite. Is $d > 0$ we are in the indefinite hyperbolic case. ■

The variable ρ is connected via an *arctan* or *arctanh* to the phase of z . But a polar decomposition is in the hyperbolic case not obvious. See [4] and notice the appearance of the Klein group.

The line $\mathcal{R}1_S$ is stabilized by σ via $z^\sigma = z \Rightarrow y = 0$. Whereas the trace maps z onto $\mathcal{R}1_S$ as

$$tr(z) = \frac{1}{2}(z + z^\sigma) = x + \frac{a}{2}y. \tag{66}$$

We define the operator spinor Mandelbrot set as

$$M := \{c|z_0 = (0, 0), \quad nr^2(z_n^c) > 0 \forall n, \quad \lim_{n \rightarrow \infty} nr^2(z_n^c) \not\rightarrow \infty.\} \tag{67}$$

In the complex case we set the parameters as

$$\begin{aligned}
a &= 0 \\
b &= -1, \quad \rightarrow \quad \gamma^2 = -1.
\end{aligned} \tag{68}$$

Hence

$$\begin{aligned}
d &= a^2 + 4b = -4 = -4\det(G) \\
\det(G) &= 1,
\end{aligned} \tag{69}$$

which results in an (anti) Euclidean geometry (Hence $e_i^2 = \pm 1, e_1^2 e_2^2 = 1$). Because of the positive discriminant we have always two roots in S . The norm yields

$$nr_{\mathbb{C}}^2(z) = x^2 + y^2. \tag{70}$$

We are left with the ordinary Mandelbrot set as shown in figure 1. All pictures with $d = -4.0$, and arbitrary a are isomorphic without a rescaling. But the coordinate interpretation is no longer possible. If one looks at $d = -4.0$, $a = 1.0$ (figure 2), one obtains up to an $\mathbf{SO}(2)$ transformation the bilinear form

$$B = G + F \cong \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}. \tag{71}$$

Thus we have now

$$\begin{aligned} \gamma e_1 &= (e_2 e_1 - B_{21})e_1 = e_1^2 e_2 = e_2 \\ \gamma e_2 &= \frac{1}{2}e_2 - l e_1. \end{aligned} \tag{72}$$

The new transformation obtained by $\gamma : \mathbf{E}(2) \rightarrow \mathbf{E}(2)$ does not preserve angles, but areas. The real axis (σ invariant points) is not affected by this. The result is the deformed set of figure 2.

If the parameter d is changed, this results in a scaling of the γ -axis. As in the above case the transition into the vector space is not unique. If one chooses the 1_S -axis to map onto e_1 then we arrive at a bilinear form like

$$[B] = \begin{pmatrix} d & \frac{a}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} G_{11} & F_{12} \\ 0 & 1 \end{pmatrix}, \tag{73}$$

which is a special case of the parametrization

$$[B] = d \begin{pmatrix} \lambda & \frac{a}{2} \\ 0 & \frac{1}{\lambda} \end{pmatrix}. \tag{74}$$

This case is also isomorphic to the complex one, but this time with an additional rescaling (figure 3). Areas are no longer preserved.

The second case is the dual one. Hence the algebra S is no longer separable and degenerated to a 1-dim. scheme. The corresponding parameters are

$$\begin{aligned} a &= 0 \\ b &= 0, \quad \rightarrow \quad \gamma^2 = 0. \end{aligned} \tag{75}$$

The discriminant vanishes. This case ($X^2 = 0$ as ideal) results in a degeneration of the dynamics. It is the limiting case of the two other ones. The semidefinite norm is $nr^2(z) = (x - a/2y)^2$, which is sensitive only to one direction. In the case with $a = 0$ this is the x -axis, see figure 4. To form a definite

norm, and thus a physically meaning full situation, one has to factor out the superfluous direction. So this case is essential one dimensional. This can take place even if B is non degenerate, but G is still.

$$[B] = \begin{pmatrix} G_{11} & F_{12} \\ -F_{12} & 0 \end{pmatrix} = \begin{pmatrix} G_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12} \\ -F_{12} & 0 \end{pmatrix} \quad (76)$$

In a physical context this case should be called trivial, but this has not to be confused with the classification of the quadratic algebras, where this case is the dual one.

The hyperbolic (or ‘perplex’) case is obtained with the parameter setting

$$\begin{aligned} a &= 0 \\ b &= 1, \quad \rightarrow \quad \gamma^2 = 1. \end{aligned} \quad (77)$$

The discriminant becomes

$$\det(G) = -1, \quad (78)$$

which corresponds to the hyperbolic geometry. Not every element in S has a root in S and especially γ has not.

In [1] the convergence was proved with the norm $nr_{\mathcal{C}}$ from above. With our considerations we get contrary

$$nr_{\mathbf{R} \oplus \mathbf{R}}^2 = x^2 - y^2. \quad (79)$$

We recover the ‘light cone’, which one is used to find in hyperbolic geometry. Hence x is the ‘time-like’ coordinate and y the ‘space-like’. Backward and forward light cone enclose the invariant real line $\mathbf{R}1_S$.

As exposed in the introduction, the dynamic (iteration process) does not respect this structure. So timelike elements will become space like and vice versa. The pictures were done in such a way, that the iteration halted immediately whenever an element got space like.

The most surprising effect is, that the light cones become separated by the multi quadratic counter part of the Mandelbrot set. The two light cones are separated thus by a timelike distance. On the real axis $\mathcal{R}1_S$ things don’t change at all. See figure 5.

The mono quadratic case would be reobtained if one would ignore the hyperbolic structure.

The hyperbolic case is the most interesting one, because of the difference to the sets obtained in literature. The asymptotic is as in the usual case. The

separation results from a deformation of the backward cone light (negative abscissa) and a minor deformation of the forward cone. If the picture is scaled in such a way, that the separation distance is small, one obtains the ordinary cone structure. Points near the space like region in the backward cone become space like during the iteration. Points inside quadrangles which intersect the real line are non divergent points and thus the counter part to the Mandelbrot set. The vertical cones without structure constitutes the space like region.

Why is this dynamically interesting case called trivial?

This stems from the quadratic relation

$$X^2 - cX = 0. \tag{80}$$

Let us assume that $c = 1$,³ then one arrives at

$$X^2 = X, \tag{81}$$

which is a projector equation. The algebra may then be decomposed with help of X into a direct sum.

$$\begin{aligned} 1_S &= X + (1 - X) \\ X(1 - X) &= 0 \\ X^2 = X &\quad (1 - X)^2 = (1 - X) \end{aligned} \tag{82}$$

Therefore $X, (1 - X)$ are pairwise annihilating primitive idempotents.

The metric structure is then connected with

$$[B] = [G] + [F] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{c}{2} \\ \frac{-c}{2} & 0 \end{pmatrix}, \tag{83}$$

which is a symplectic structure or in an alternative way isomorphic⁴ to

$$[B] = [G] = \begin{pmatrix} \frac{c}{2} & 0 \\ 0 & -\frac{c}{2} \end{pmatrix}, \tag{84}$$

without any antisymmetric contribution! By such a change, we deal with isomorphic quadratic algebras. The underlying Clifford algebras are no longer

³ (The sign does not matter in this case)

⁴ With the isomorphism criterion from section four. Use the substitution $X' = aX + b$, with $a = 2$ and $b = -c$. This transformation is not grade preserving and thus super symmetric.

isomorphic, because we changed the symmetric part of the bilinearform and thus the involved quadratic form. A change of this kind prevent us from a geometric interpretation. The quadratic algebra is less restrictive than the Clifford algebra, which is needed for a geometrical interpretation in the (grade one) vector space. This corresponds to $X^2 = c^2 > 0$. The decomposition is now obtained by the projectors

$$e_{\pm} := \frac{1}{2c}(c \pm X) \quad (85)$$

The parameter a acts as in the complex case, which is seen in figure 6. The fact that the trivial case can always be splitted into a direct sum with diagonal multiplicative structure was essential for the proof in [1].

In the pure hyperbolic case ($d = 4$, $a = 0$) we have for example

$$[B] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (86)$$

and

$$\begin{aligned} \gamma e_1 &= e_2 \\ \gamma e_2 &= e_1, \end{aligned} \quad (87)$$

which is not a rotation, but a ‘space-time’ inversion. This transformation flips also the orientation of \mathcal{V} , thus a physical interpretation should be charge or parity conjugation. But there is a continuum of such transformations.

6. Conclusion.

We showed with numerical examples, that the multi quadratic Mandelbrot set is superior to the quadratic one. The geometric interpretation fits in all special cases, but then with a distinguished (pseudo) norm. The operator spinor approach is the key step in this consideration. In a first step we considered the two dimensional case, which has to be enlarged to higher dimensions. Thereby the theorems from [14] provide us that the same structure appears as tensor factor in the representation theory of Clifford Algebras.

The strong connection between conjugation and (pseudo) norm, as with the geometry of the underlying vector space (module...) was shown. Thus a knowledge of the bilinear form in \mathcal{V} provides us with all information needed. One is able to choose a special representant out of an ideal consisting of isomorphic such representants. The dependence of the multivector structure on these choices was shown.

The field \mathbf{C} plays a special role, as the only one with a single nontrivial isomorphism class. This may be the origin of the usefulness of the complex numbers in quantum mechanics and nonlinear classical mechanics. This fact remains in higher dimensions.

We remarked the richness of this structure if the underlying space is build up over rings as \mathbf{Z} or finite fields as \mathbf{F}_q . An investigation in this direction should result in much more different cases. These cases are quite interesting in quantum theory, because they will be expected to be connected with inequivalent representations. Normally this is achieved only with infinitely many particles.

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Appendix

The figures are calculated with 800×800 points resolution and 500 iterations in a window $[-5 : 5]$ for the 1_S (hor.) and γ (vert.) axis. If the norms got negative, the iteration had been stopped. The potential lines give the tendency of reaching infinity by surmounting a given threshold in n -steps. Twelve such lines are plotted. The interior of the Mandelbrot set and the multi quadratic set consists of non divergent points. $d = a^2 + 4b$ is the discriminant. a is as in the text the linear part of the quadratic relation.

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