

CLIFFORD ALGEBRA AND THE LORENTZ TRANSFORMATION WITH (p, q) FORM

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Abstract. In this paper, the concepts of Lorentz inner product with (p, q) form, the Lorentz space and the Lorentz transformation with (p, q) form are given by using Clifford algebra. It is shown that $L_m^{p,q}$ is the Lorentz transformation with (p, q) form, and the matrix equality relation of Minkowski space with $(n - 1, 1)$ form is given. The examples are given to illustrate the corresponding results.

Key words: Clifford Algebra, Lorentz Transformation, Lorentz Transformation with (p, q) form.

1. Introduction

Clifford algebra was founded by the British mathematician W.K. Clifford in 1878. It has been applied widely in maths, physics and engineering technology fields etc. In this paper, by means of the concepts of Lorentz inner product with (p, q) form, the Lorentz space and the Lorentz transformation with (p, q) form, $L_m^{p,q}$ is considered. The result shows that $L_m^{p,q}$ is the Lorentz transformation with (p, q) form. The corresponding results in references [2]-[5] become the special cases in this paper.

2. Definitions

Definition 2.1 Define a real function with two variables on the vector-plane \mathbf{R}^n , and denote $\langle \mathbf{a}, \mathbf{b} \rangle_L$. $\langle \mathbf{a}, \mathbf{b} \rangle_L$ is the Lorentz inner product with (p, q) form on the vector-plane \mathbf{R}^n , if the following properties hold, that is

- (1) $\langle \mathbf{a}, \mathbf{b} \rangle_L = \langle \mathbf{b}, \mathbf{a} \rangle_L$
- (2) $\langle \lambda \mathbf{a}, \mathbf{b} \rangle_L = \lambda \langle \mathbf{a}, \mathbf{b} \rangle_L$
- (3) $\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle_L = \langle \mathbf{a}, \mathbf{c} \rangle_L + \langle \mathbf{b}, \mathbf{c} \rangle_L$
- (4) there exist base-vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = \begin{cases} 1, & i = j \leq p; \\ -1, & p < i = j \leq p + q; \\ 0, & i \neq j. \end{cases}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^n, \lambda \in \mathbf{R}, 0 \leq p, q \leq n, p + q = n$.

Definition 2.2 A space is defined as Lorentz space, if it keeps the Lorentz inner product.

Definition 2.3 For an arbitrary vector $\alpha = (x_1, x_2, \dots, x_p, x_{p+1}, \dots, x_n) \in \mathbf{R}^n$, the transformation that keeps $x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2$ is called Lorentz transformation with (p, q) form.

3. $Cl_{p,q}(n = p + q)$ and the Lorentz Transformation with (p, q) Form

A basic vector system $e_1, e_2, \dots, e_p, e_{p+1}, \dots, e_n$ of Clifford algebra $Cl_{p,q}(n = p + q)$, under the Clifford product and dot product satisfy:

$$\mathbf{e}_i \mathbf{e}_j = \begin{cases} 1, & 1 < i = j \leq p; \\ -1, & p < i = j \leq n; \\ -\mathbf{e}_j \mathbf{e}_i, & i \neq j, \end{cases}$$

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = \begin{bmatrix} E_p & O \\ O & -E_q \end{bmatrix}.$$

Create an n -dimensional ($n = p + q$) Minkowski space with (p, q) form. The space is denoted by $M_{p,q} = \{\mathbf{r} + \mathbf{b}\}$, for $\mathbf{r} = (x_1, x_2, \dots, x_p)$ and $\mathbf{b} = (x_{p+1}, x_{p+2}, \dots, x_n)$.

Definition 3.1 An arbitrary vector $\mathbf{w} = \mathbf{r} + \mathbf{b} \in M_{p,q}$, its conjugate vector is defined as $\mathbf{w}^* = \mathbf{r} - \mathbf{b}$.

Let the multiplication operation over $M_{p,q}$ be

$$(\mathbf{r}_1 + \mathbf{b}_1) \cdot (\mathbf{r}_2 + \mathbf{b}_2) = x_1 y_1 + \dots + x_p y_p - x_{p+1} y_{p+1} - \dots - x_n y_n$$

According to its multiplication operation, the norm of vector \mathbf{w} is defined as

$$\sigma(\mathbf{w}) = |\mathbf{w} \cdot \mathbf{w}|^{\frac{1}{2}}$$

Let $M_{p,q}^+ = \{\mathbf{r} + \mathbf{b} | \sigma(\mathbf{b}) \geq \sigma(\mathbf{r})\}$, $M_{p,q}^+$ is said to be a future timelike region of $M_{p,q}^+$.

Define a multiplication operation on $M_{p,q}^+$:

$$\circ : \mathbf{w}_1 \circ \mathbf{w}_2 = (\mathbf{r}_1 + b_1) \circ (\mathbf{r}_2 + b_2) = \sigma(\mathbf{b}_1)\mathbf{r}_2 + \sigma(\mathbf{b}_2)\mathbf{r}_1 + (\mathbf{r}_1 \cdot \mathbf{r}_2 + \sigma(\mathbf{b}_1)\sigma(\mathbf{b}_2))\mathbf{E}$$

for \mathbf{E} is a q -dimensional unit vector, then $M_{p,q}$ is an n -dimensional real algebra, but it is not space-time algebra under the operation \circ .

Hence, by using the norm and the binary operation \circ , we introduce a binary operation over $M_{p,q}$ as that follows:

$$\odot : \mathbf{w}_1 \odot \mathbf{w}_2 = (\mathbf{r}_1 + b_1) \odot (\mathbf{r}_2 + b_2) = (\mathbf{r}_1 + b_1) \circ (\mathbf{r}_{2\parallel} + b_2) + \sigma(\mathbf{w}_1)\mathbf{r}_{2\perp}$$

Theorem 3.2 For all $\mathbf{m} \in M_{p,q}^+ (n = p + q)$, let

$$L_m^{p,q} : M_{p,q}^+ \rightarrow M_{p,q}^+, \mathbf{m} \mapsto -\mathbf{m}^* \odot \mathbf{w}$$

for $\sigma(\mathbf{m}) = 1$, then $L_m^{p,q}$ is the Lorentz transformation with (p, q) form of Minkowski space with (p, q) form.

Proof: Denote $\mathbf{m} = \mathbf{r} + \mathbf{b}$, $\mathbf{w} = \mathbf{r} + \mathbf{b}_1$, $\mathbf{w}' = \mathbf{r} + \mathbf{b}'$.

Let $\mathbf{w}' = -\mathbf{m}^* \odot \mathbf{w}$, for all $\mathbf{m}, \mathbf{w}, \mathbf{w}' \in M_{p,q}^+, \sigma(\mathbf{m}) = 1$, by the definition of \odot , we obtain

$$\mathbf{r}' = \sigma(\mathbf{b})\mathbf{r}_{\parallel} - \sigma(\mathbf{b}_1)\mathbf{r}_{\mathbf{m}} + \mathbf{r}_{\perp},$$

$$\mathbf{b}' = (\sigma(\mathbf{b})\sigma(\mathbf{b}_1) - \mathbf{r}_{\mathbf{m}} \cdot \mathbf{r}_{\parallel})\mathbf{E}$$

Since $\mathbf{r}_{\parallel} = (\mathbf{r}_{\mathbf{m}} \cdot \mathbf{r})\mathbf{r}/r_{\mathbf{m}}^2, \mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{\parallel}$, then the Lorentz transformation with (p, q) form holds. That is

$$(\mathbf{b}')^2 - (\mathbf{r}')^2 = \mathbf{b}_1^2 - \mathbf{r}^2.$$

4. Examples

Example 4.1 When $p = q = 1$, for any $\mathbf{m} \in M_2^+ = \{x + jct \in M_2 \mid ct \geq |x|\}$, if and only if when $x = 0$, the equality sign is correct, then $L_m^{1,1}$ is the Lorentz transformation with $(1, 1)$ form.

Example 4.2 When $p = 3, q = 1$, for any $\mathbf{m} \in M_4^+ = \{\mathbf{r} + jct \in M_4 \mid ct \geq \sigma(\mathbf{r})\}$, then $L_m^{3,1}$ is the Lorentz transformation with $(3, 1)$ form. This transformation is Lorentz transformation in physical problems of special relatively.

Example 4.3 When $p = n - 1, q = 1$, for any $\mathbf{m} \in M_n^+ = \{\mathbf{r} + jct \in M_n \mid ct \geq \sigma(\mathbf{r})\}$, then $L_m^{n-1,1}$ is the Lorentz transformation with $(n - 1, 1)$ form.

Proof: Denote $\mathbf{m} = \mathbf{r}_m + jct, \mathbf{w} = \mathbf{r} + jct_1, \mathbf{w}' = \mathbf{r}' + jct'_1$.

Let $\mathbf{w}' = -\mathbf{m}^* \odot \mathbf{w}$, for all $\mathbf{m}, \mathbf{w}, \mathbf{w}' \in M_n^+$.

By $\mathbf{w}_1 \odot \mathbf{w}_2 = (\mathbf{r}_1 + jct_1) \odot (\mathbf{r}_2 + jct_2) + \sigma(\mathbf{w}_1)\mathbf{r}_{2\perp}$

we obtain

$$\mathbf{r}' = ct\mathbf{r}_{\parallel} - ct_1\mathbf{r}_{\mathbf{m}} + \mathbf{r}_{\perp}, ct_1 = c^2tt_1 - \mathbf{r}_m \cdot \mathbf{r}_{\parallel}$$

then we have

$$(ct'_1)^2 - (\mathbf{r}')^2 = (ct_1)^2 - \mathbf{r}^2.$$

Similar to the proof of Example 3, Example 2 can be obtained.

If we denote $\mathbf{m} = j(ch\theta + jrsh\theta)$, by $\mathbf{r}_m^0 = \frac{\mathbf{r}_m}{r_m}, r = (1 - \frac{v^2}{c^2})^{\frac{1}{2}}$, we obtain

$$\mathbf{r}' = r(\mathbf{r}_{\parallel} - \mathbf{v}t) + \mathbf{r}_{\perp}, t' = r(t - \frac{\mathbf{v}\mathbf{r}}{c^2})$$

Let $\mathbf{v} = (v_1, v_2, \dots, v_{n-1})$, then we have

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