COMMUTATIVE HYPERCOMPLEX NUMBERS AND FUNCTIONS OF HYPERCOMPLEX VARIABLE: A MATRIX STUDY

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Abstract. Systems of hypercomplex numbers, which had been studied and developed at the end of the 19th century, are nowadays quite unknown to the scientific community. It is believed that study of their applications ended just before one of the fundamental discoveries of the 20th century, Einstein's equivalence between space and time. Owing to this equivalence, not-defined quadratic forms have got concrete physical meaning and have been recently recognized to be in strong relationship with a system of bidimensional hypercomplex numbers. These numbers (called *hyperbolic*) can be considered as the most suitable mathematic language for describing the bidimensional space-time, in spite of some unfamiliar algebraic properties common to all the commutative hypercomplex systems with more than two dimensions: they are decomposable systems and there are non-zero numbers whose product is zero. With respect to the famous Hamilton quaternions, one can introduce the differential calculus for the hyperbolic numbers and for all the commutative hypercomplex systems; moreover, one can even introduce functions of hypercomplex variable.

The aim of this work is to study the systems of commutative hypercomplex numbers and the functions of hypercomplex variable by describing them in terms of a familiar mathematical tool, i.e. matrix algebra.

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Contents

1. Introduction

It is well known that the application of the complex numbers to the solution of scientific problems goes beyond their algebraic introduction. As examples, we cite the use of complex variables for representing vectors in a plane and the use of functions of a complex variable to solve Laplace's equation [1]. Hamilton introduced non-commutative quaternions for representing vectors in the three-dimensional space [2].

As far as hypercomplex numbers are concerned, we can introduce functions just for commutative systems [3]. In these systems, however, there are non-zero numbers whose product is zero [3], a property which is not found for real and complex numbers. Moreover, systems of hypercomplex numbers can be decomposed into subsystems of real and complex numbers [3]. For these reasons, the theory of functions of hypercomplex numbers was not developed. However, in modern times it was recognized that a special kind of hypercomplex numbers – the hyperbolic numbers – can be regarded as the "mathematics of Special Relativity" [4, 5] and a physical meaning has been attributed to functions of hyperbolic variable [6]. Such applications of hyperbolic numbers, and other ones [7, 8], make us thrust that similar applications can be found for commutative hypercomplex numbers [9].

The aim of this paper is to study hypercomplex systems by using matrix algebra, which is well known to the scientific community. This formalism not only allows finding the algebraic properties of hypercomplex numbers, but it also lets us to introduce the theory of functions and differential calculus for hypercomplex systems.

The paper is organized in the following way: in Sections 2.1 and 3, the matrix formalism is associated with hypercomplex numbers. In Section 3, the algebraic properties of systems of hypercomplex numbers are outlined. In Sections 4 and 5, the functions of hypercomplex variable and the differential calculus are eventually introduced.

2. Mathematical Operations

In order to build a coherent theory of hypercomplex numbers based on their matrix representation, let us start with some important definitions and properties relative to the mathematical operations among these numbers.

2.1. Preliminary considerations

Let us introduce an N-dimensional hypercomplex number as a column vector, whose entries are the components of the number. If $\mathbf{x}$ is the hypercomplex number [3]:

$$\mathbf{x} = \sum_{k=0}^{N-1} x_k \mathbf{e}_k \,, \tag{2.1}$$

where $x_k \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) are called the components of $\mathbf{x}$ and $\mathbf{e}_k$ are the unit vectors along the N dimensions. In matrix representation, we can write $\mathbf{x}$ in its matrix form

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \tag{2.2}$$

and the unit vectors, that we will call *natural* unit vectors, as

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \dots, \qquad \mathbf{e}_{N-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Equation (2.2) defines the matrix representation of $\mathbf{x}$.

Hereafter, when unnecessary, we will not distinguish between hypercomplex numbers and their matrix representations. Moreover, we will label the set of N-dimensional hypercomplex numbers with the symbol $\mathbb{H}_N$ and use the term $\mathbb{H}$-number as a shorthand to hypercomplex number.

2.2. Equality, sum, and scalar multiplication

2.2.1. Equality of two hypercomplex numbers

Two $\mathbb{H}$-numbers are equal if all their components are equal, one by one. One can verify that such a definition is an equivalency relationship, because it satisfies the reflexive, symmetric, and transitive properties.

2.2.2. Sum and difference of two hypercomplex numbers

As in the for column vectors, the sum of two $\mathbb{H}$-numbers is defined by summing their components. The result is an $\mathbb{H}$-number.

By exploiting the above introduced matrix representation, one can verify that the sum operation is both commutative and associative. The column vector with all zero components, indicated with $\mathbf{0}$, is the null element for the sum, which we name as *zero*. With respect to the sum, the inverse element of $\mathbf{x}$ is $-\mathbf{x}$, which is defined as having all its components changed in sign with respect to the components of $\mathbf{x}$. Therefore, $(\mathbb{H}_N, +)$ is an Abelian group.

In analogy with the above definition of sum, the definition of difference between $\mathbb{H}$-numbers can be stated as the difference between column vectors.

2.2.3. Multiplication of a real number by a hypercomplex number

In agreement with the multiplication of a scalar by a matrix, the multiplication of $a \in \mathbb{R}$ by $\mathbf{x} \in \mathbb{H}_N$ is an $\mathbb{H}$-number, whose components are the components of $\mathbf{x}$ multiplied by a, one by one. Because of commutativity in the multiplication of two real numbers, the product of a by $\mathbf{x}$ can be indicated with $a\,\mathbf{x}$ or $\mathbf{x}\,a$.

2.3. Product and related operations

2.3.1. Multiplication of two hypercomplex numbers

Before defining the multiplication between $\mathbb{H}$-numbers, let us introduce the multiplication of the natural unit vectors $\mathbf{e}_k$ $(k = 0, 1, ..., N-1)$ by any $\mathbf{y} \in \mathbb{H}_N$ as

$$\mathbf{e}_k \cdot \mathbf{y} = \mathcal{A}_k \mathbf{y}, \tag{2.3}$$

where $\mathcal{A}_k$ is a proper $N \times N$ matrix, and the operation in the right-hand side of Eq. (2.3) is the well known multiplication of a square matrix by a column vector. Note that $\mathcal{A}_k \mathbf{y}$ is a column vector with real components, therefore it is a $\mathbb{H}$-number as well. The representation introduced in Eq. (2.3) of the product of a unit vector for a hypercomplex number gives the same results as the multiplication by means of the structure constants [3] if the following relation holds for all the elements of the matrices $\mathcal{A}_k$:

$$(\mathcal{A}_k)_{ij} = C^i_{kj}.$$

The multiplication between two $\mathbb{H}$-numbers, $\mathbf{x}$ and $\mathbf{y}$, can be conveniently defined by applying Eq. (2.3) to Eq. (2.1). It ensues:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=0}^{N-1} x_k \mathbf{e}_k \cdot \mathbf{y} = \sum_{k=0}^{N-1} x_k \left(\mathcal{A}_k \mathbf{y} \right) . \tag{2.4}$$

One can verify that if

$$(\mathcal{A}_k)_{ij} = (\mathcal{A}_j)_{ik} , \tag{2.5}$$

is true, then the multiplication between $\mathbb{H}$-numbers, defined with Eq. (2.4), is commutative. Hereafter, we will consider only multiplication-commutative systems of $\mathbb{H}$-numbers, for which Eq. (2.5) is true.

Equation (2.4) can also be written as

$$\mathbf{x} \cdot \mathbf{y} = \mathcal{M}(\mathbf{x}) \mathbf{y} , \tag{2.6}$$

where the matrix $\mathcal{M}(\mathbf{x})$ – whose dependence on $\mathbf{x}$ has to be understood as an explicit dependence on the components[1] of $\mathbf{x}$, and we write $\mathcal{M}(\mathbf{x})$ instead of $\mathcal{M}(x_0, x_1, \ldots, x_{N-1})$ for the sake of brevity – is defined as

$$\mathcal{M}(\mathbf{x}) = \sum_{k=0}^{N-1} x_k \mathcal{A}_k \tag{2.7}$$

[1] That is, $\mathcal{M}$ is not a function of $\mathbf{x}$, but is a function of its components.

and is called *associated matrix* of the hypercomplex number $\mathbf{x}$. The matrix $\mathcal{M}(\mathbf{x})$ is the transposed matrix of the characteristic matrix considered in Ref. [3].[2] Because Eq. (2.5) is assumed as true, one can alternatively write

$$\mathbf{x} \cdot \mathbf{y} = \mathcal{M}(\mathbf{y})\mathbf{x} \tag{2.8}$$

instead of Eq. (2.6).

Now, let us state the conditions for existence of the *neutral element* for the multiplication between $\mathbb{H}$-numbers. We will name such an element as the *unity* of $\mathbb{H}_N$, and indicate it with the symbol $\mathbf{1}$. By definition, for any $\mathbf{x} \in \mathbb{H}_N$,

$$\mathbf{1} \cdot \mathbf{x} = \mathbf{x} \tag{2.9}$$

must hold true. Therefore, if 1_k are the components of $\mathbf{1}$, by applying Eq. (2.4) one finds

$$\sum_{k=0}^{N-1} 1_k \mathcal{A}_k \mathbf{x} = \mathbf{x}\,. \tag{2.10}$$

Because Eq. (2.10) must hold true for *any* $\mathbf{x}$, the following condition of existence of the $\mathbb{H}_N$ unity is found:

$$\sum_{k=0}^{N-1} 1_k \mathcal{A}_k = \mathcal{I}\,, \tag{2.11}$$

where $\mathcal{I}$ is the identity $N \times N$ matrix. Hereafter, we will assume that this condition of existence of the $\mathbb{H}_N$ unity is satisfied – we will see later, with some examples, that the unity actually exists for the $\mathbb{H}_N$ systems we will consider.

Now, let us demonstrate that the unity $\mathbf{1}$ is unique.

Proof – Let us suppose that another unity, $\mathbf{1}'$, exists in $\mathbb{H}_N$. In such a case, both

$$\mathbf{1} \cdot \mathbf{1}' = \mathbf{1}' \text{ and } \mathbf{1}' \cdot \mathbf{1} = \mathbf{1}$$

should hold true. Because of multiplication commutativity, one concludes that $\mathbf{1}' = \mathbf{1}$. Thus, it is demonstrated that $\mathbf{1}$, if it exists, is unique. □

[2] Let us explain the reason for these different formalisms. In Ref. [3], the characteristic matrix has been introduced in the study of the linear system defining the division. As a consequence, the hypercomplex numbers must be row vectors. In this paper, the hypercomplex numbers have been introduced, following the convention of linear algebra, as column vectors. Thus, the associated matrix is the transpose of the characteristic matrix.

2.3.2. Inverse element for the multiplication

Given a $\mathbf{x} \in \mathbb{H}_N$, the *inverse element* for the multiplication is that $\mathbb{H}$-number, $\mathbf{x}^{-1}$, that satisfies

$$\mathbf{x} \cdot \mathbf{x}^{-1} = \mathbf{1}\,. \tag{2.12}$$

Since

$$\mathbf{x} \cdot \mathbf{x}^{-1} = \mathcal{M}(\mathbf{x})\mathbf{x}^{-1}\,,$$

the inverse element exists only if the determinant of $\mathcal{M}(\boldsymbol{x})$ – let us call it the characteristic determinant [3] – is non-zero, and one gets

$$\mathbf{x}^{-1} = \mathcal{M}^{-1}(\mathbf{x})\mathbf{1}\,, \tag{2.13}$$

where $\mathcal{M}^{-1}(\mathbf{x})$ is the matrix inverse of $\mathcal{M}(\mathbf{x})$. Because $\mathbf{1}$ is unique and the matrix inverse is univocally defined, Eq. (2.13) states also that if the inverse element exists, it is unique.

Equation (2.13) implies that, for *any* $\mathbf{y} \in \mathbb{H}_N$, the following equation holds

$$\mathbf{x}^{-1} \cdot \mathbf{y} = \mathcal{M}^{-1}(\mathbf{x})\mathbf{y}\,.$$

Therefore,

$$\mathcal{M}(\mathbf{x}^{-1}) = \mathcal{M}^{-1}(\mathbf{x}). \tag{2.14}$$

Moreover, because Eq. (2.12) leads to

$$\left(\mathbf{x}^{-1}\right)^{-1} = \mathbf{x}\,,$$

by taking into account Eq. (2.14) one gets

$$\mathcal{M}^{-1}(\mathbf{x}^{-1}) = \mathcal{M}(\mathbf{x})\,.$$

One further property is

$$(\mathbf{x} \cdot \mathbf{y})^{-1} = \mathbf{x}^{-1} \cdot \mathbf{y}^{-1}.$$

Taking into account commutativity, this relation is the same one that holds for square matrices in matrix algebra.

Furthermore, by applying Eq. (2.12) one can verify that the inverse element of $\mathbf{1}$ is $\mathbf{1}$ itself, i.e.

$$\mathbf{1}^{-1} = \mathbf{1}.$$

2.3.3. Other properties of the multiplication

The multiplication between $\mathbb{H}$-numbers satisfies the distributive property with respect to the sum, i.e.

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}=\mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}\,.$$

Proof – As a matter of fact, the k^{th} component of $\mathbf{x}+\mathbf{y}$ being equal to x_k+y_k, one gets from Eq. (2.4)

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}=\sum_{k=0}^{N-1}(x_k+y_k)\,\mathcal{A}_k\mathbf{z}=\sum_{k=0}^{N-1}x_k\mathcal{A}_k\mathbf{z}+\sum_{k=0}^{N-1}y_k\mathcal{A}_k\mathbf{z}=\mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}\,.\quad\Box$$

Now let us demonstrate a fundamental property: the commutative $\mathbb{H}$-numbers are associative.
Proof – As a matter of fact, given $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{H}_N$, by taking into account commutativity and the above-demonstrated distributive property, it ensues

$$(\mathbf{x}\cdot\mathbf{y})\cdot\mathbf{z}=\mathbf{z}\cdot(\mathbf{x}\cdot\mathbf{y})=\mathbf{z}\cdot\left(\sum_{k=0}^{N-1}x_k\mathcal{A}_k\mathbf{y}\right)=\sum_{k=0}^{N-1}x_k\mathbf{z}\cdot\mathcal{A}_k\mathbf{y}=\sum_{k,n=0}^{N-1}x_kz_n\mathcal{A}_n\mathcal{A}_k\mathbf{y}\,,$$

$$\mathbf{x}\cdot(\mathbf{y}\cdot\mathbf{z})=\mathbf{x}\cdot(\mathbf{z}\cdot\mathbf{y})=\mathbf{x}\cdot\left(\sum_{n=0}^{N-1}z_n\mathcal{A}_n\mathbf{y}\right)=\sum_{n=0}^{N-1}z_n\mathbf{x}\cdot\mathcal{A}_n\mathbf{y}=\sum_{k,n=0}^{N-1}x_kz_n\mathcal{A}_k\mathcal{A}_n\mathbf{y}\,.$$

Therefore, because the matrices $\mathcal{A}_k$ satisfy

$$\mathcal{A}_k\mathcal{A}_n=\mathcal{A}_n\mathcal{A}_k \tag{2.15}$$

for any $k,n=0,1,...,N-1$, as it follows from $\mathbf{e}_k\cdot\mathbf{e}_n=\mathbf{e}_n\cdot\mathbf{e}_k$, the associative property

$$(\mathbf{x}\cdot\mathbf{y})\cdot\mathbf{z}=\mathbf{x}\cdot(\mathbf{y}\cdot\mathbf{z})$$

is demonstrated. □

Another property that can be easily verified is

$$\mathbf{x}\cdot\mathbf{0}=\mathbf{0}.$$

2.3.4. Division between hypercomplex numbers

Given two $\mathbb{H}$-numbers, $\mathbf{x}$ and $\mathbf{y}$, with $\det[\mathcal{M}(\mathbf{y})]\neq 0$, let us define the division of $\mathbf{x}$ by $\mathbf{y}$ as the product of $\mathbf{x}$ by the inverse element of $\mathbf{y}$, i.e.

$$\frac{\mathbf{x}}{\mathbf{y}}\equiv\mathbf{x}\cdot\mathbf{y}^{-1}\,. \tag{2.16}$$

It is easy to verify that

$$\frac{\mathbf{x}}{\mathbf{1}} = \mathbf{x}, \quad \frac{\mathbf{1}}{\mathbf{y}} = \mathbf{y}^{-1}.$$

Moreover, as one can verify, many of the rules of fraction algebra of real numbers can be extended to $\mathbb{H}$-numbers. Among them, we report:

$$\frac{\mathbf{1}}{\mathbf{x}/\mathbf{y}} = \frac{\mathbf{y}}{\mathbf{x}};$$

$$\frac{\mathbf{x}_1}{\mathbf{y}_1} \cdot \frac{\mathbf{x}_2}{\mathbf{y}_2} = \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_2};$$

$$\frac{\mathbf{x}_1}{\mathbf{y}_1} + \frac{\mathbf{x}_2}{\mathbf{y}_2} = \frac{\mathbf{x}_1 \cdot \mathbf{y}_2 + \mathbf{x}_2 \cdot \mathbf{y}_1}{\mathbf{y}_1 \cdot \mathbf{y}_2}.$$

In the above identities, the non-singularity of the associated matrices with $\mathbb{H}$-numbers that appear in the denominators is assumed.

2.3.5. Divisors of zero

Before, we have seen that the inverse element of $\mathbf{y}$ for the multiplication is not defined if $\det[\mathcal{M}(\mathbf{y})] = 0$ and that, in such a case, the ratio $\mathbf{x}/\mathbf{y}$ cannot be defined. An $\mathbb{H}$-number $\mathbf{y} \neq \mathbf{0}$, whose associated matrix $\mathcal{M}(\mathbf{y})$ is singular, is named *divisor of zero.*

The reason for such a name is explained in the following. Let us consider, in the domain of $\mathbb{H}$-numbers, the equation

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{0} \tag{2.17}$$

in the unknown $\mathbf{x}$, where $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and $\mathbf{y}$ is a divisor of zero. The equation can be rewritten in its equivalent matrix form:

$$\mathcal{M}(\mathbf{y})\mathbf{x} = \mathbf{0}. \tag{2.18}$$

In matrix language, the above Eq. (2.18) represents a homogeneous system of N equations with N unknowns, $x_0, x_1, ..., x_{N-1}$. Because $\mathcal{M}(\mathbf{y})$ is singular, i.e. its determinant is zero, it is a well known result that the above system of equations admits infinite non-zero solutions $\mathbf{x}$. Stated in $\mathbb{H}$-number terms, because $\mathbf{y}$ is a divisor of zero, a numerable infinity of non-zero $\mathbb{H}$-numbers $\mathbf{x}$ exists that satisfy Eq. (2.17). This means that two (proper) *non*-zero $\mathbb{H}$-numbers can have zero product, something that never happens for real or complex numbers.

In such a case, we will say that the two $\mathbb{H}$-numbers, $\mathbf{x}$ and $\mathbf{y}$, are *orthogonal*. The set of non-zero solutions $\mathbf{x}$ to Eq. (2.17), indicated as $\perp_{\mathbf{y}}$, will be named as the *orthogonal set* of $\mathbf{y}$. Equation (2.17) can also be formally written as

$$\mathbf{x} = \frac{\mathbf{0}}{\mathbf{y}} \neq \mathbf{0}\,,$$

from which the reason of the name *divisor of zero* is explained for $\mathbf{y}$.

Formally, Eq. (2.17) could also be written as

$$\mathbf{y} = \frac{\mathbf{0}}{\mathbf{x}}\,.$$

One can wonder if the above equation makes sense. If it does, a direct implication is that $\mathbf{x}$, which is a non-zero solution of Eq. (2.17), is a divisor of zero as well.

Proof – An equivalent matrix form of Eq. (2.17) is

$$\mathcal{M}(\mathbf{x})\mathbf{y} = \mathbf{0}\,. \tag{2.19}$$

Because $\mathbf{y}$ is a divisor of zero, and therefore $\mathbf{y} \neq \mathbf{0}$, Eq. (2.19) implies $\det[\mathcal{M}(\boldsymbol{x})] = 0$, thus also $\mathbf{x}$ is a divisor of zero. This demonstrates that all the elements of $\perp_{\mathbf{y}}$ are divisors of zero. □

As a special case, let us demonstrate that $\mathbf{1}$ is *not* a divisor of zero.

Proof – If $\mathbf{1}$ were a divisor of zero, then at least one $\mathbf{x} \neq \mathbf{0}$, which is *not* a divisor of zero, would exist such that

$$\mathbf{x} \cdot \mathbf{1} = \mathbf{0}\,,$$

something that contradicts the definition of $\mathbf{1}$. □

To conclude, one can verify that if $\mathbf{x}$ is a divisor of zero and $\mathbf{y}$ is any non-zero $\mathbb{H}$-number such that $\mathbf{y} \notin \perp_{\mathbf{x}}$, then also $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x}/\mathbf{y}$ are divisors of zero (in the latter case, $\mathbf{y}$ is assumed to be not a divisor of zero).

2.3.6. Power of a hypercomplex number

Given any integer $n > 0$, the n^{th} power of the $\mathbb{H}$-number $\mathbf{x}$ is defined by multiplying $\mathbf{x}$ by itself n times. Being

$$\mathbf{x} = \mathbf{x} \cdot \mathbf{1} = \mathcal{M}(\mathbf{x})\mathbf{1}\,,$$

one gets

$$\mathbf{x}^n = \mathcal{M}^n(\mathbf{x})\mathbf{1}\,, \tag{2.20}$$

where $\mathcal{M}^n(\mathbf{x})$ indicates the power of a matrix. Conventionally, we set

$$\mathbf{x}^0 = \mathbf{1}\,.$$

Equation (2.20) can be viewed as the actual definition of power. It can be formally extended to negative-exponent powers – only for non-zero $\mathbb{H}$-numbers which are not divisors of zero – with

$$\mathbf{x}^{-n} = \left(\mathbf{x}^{-1}\right)^n = \mathcal{M}^{-n}(\mathbf{x})\mathbf{1}\,,$$

where $\mathcal{M}^{-n}(\mathbf{x})$ means

$$\mathcal{M}^{-n}(\mathbf{x}) \equiv \left[\mathcal{M}^{-1}(\mathbf{x})\right]^n\,.$$

Under proper conditions for the determinants of the involved matrices, one can verify that many of the known properties that hold true for real numbers can be extended to $\mathbb{H}$-numbers. Among them, we report:

$$\mathbf{0}^n = \mathbf{0}; \qquad \mathbf{1}^n = \mathbf{1};$$

$$(\mathbf{x}^m)^n = \mathbf{x}^{mn}; \quad \mathbf{x}^m \cdot \mathbf{x}^n = \mathbf{x}^{m+n}; \quad (\mathbf{x}\cdot\mathbf{y})^n = \mathbf{y}^n \cdot \mathbf{y}^n;$$

$$\frac{\mathbf{x}^m}{\mathbf{x}^n} = \mathbf{x}^{m-n}; \quad \left(\frac{\mathbf{x}}{\mathbf{y}}\right)^n = \frac{\mathbf{x}^n}{\mathbf{y}^n}.$$

These properties can be demonstrated by taking into account

$$\mathcal{M}(\mathbf{x}\cdot\mathbf{y}) = \mathcal{M}(\mathbf{x})\mathcal{M}(\mathbf{y}) \tag{2.21}$$

and

$$\mathcal{M}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) = \mathcal{M}(\mathbf{x})\mathcal{M}^{-1}(\mathbf{y})\,. \tag{2.22}$$

Proof – As a matter of fact, given another arbitrary $\mathbb{H}$-number, $\mathbf{z}$, by exploiting the associativity of $\mathbb{H}$-numbers one has

$$\mathcal{M}(\mathbf{x}\cdot\mathbf{y})\cdot\mathbf{z} = (\mathbf{x}\cdot\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot(\mathbf{y}\cdot\mathbf{z}) = \mathbf{x}\cdot\mathcal{M}(\mathbf{y})\mathbf{z} = \mathcal{M}(\mathbf{x})\mathcal{M}(\mathbf{y})\mathbf{z}\,,$$

from which Eq. (2.21) follows. Equation (2.22) can be similarly demonstrated. □

2.4. Bidimensional hypercomplex numbers and complex numbers

As an example, let us consider bidimensional $\mathbb{H}$-numbers. When $N = 2$, the generally defined $\mathbb{H}$-number, $\mathbf{z}$, is

$$\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.23}$$

and its associated matrix, which is the transpose of the characteristic matrix [3], is defined as

$$\mathcal{M}\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x & \alpha y \\ y & x + \beta y \end{pmatrix}, \tag{2.24}$$

where α and β are real numbers.

Now, let us verify that the bidimensional system of $\mathbb{H}$-numbers defined by Eq. (2.24) is multiplication-commutative, multiplication-associative, and that the multiplicative unity exists.

Proof – The matrices $\mathcal{A}_0$ and $\mathcal{A}_1$, associated with the natural unity vectors, $\mathbf{e}_0$ and $\mathbf{e}_1$, can be obtained from Eq. (2.24) by setting in it $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$, respectively. They are

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_1 = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}. \tag{2.25}$$

It can be easily verified that $\mathcal{A}_0$ and $\mathcal{A}_1$ satisfy Eq. (2.5), thus the commutativity of multiplication is demonstrated. Moreover, since $\mathcal{A}_0 \equiv \mathcal{I}$, where $\mathcal{I}$ is the 2×2 identity matrix, Eq. (2.15) is satisfied, thus the multiplication is also associative. Regarding the multiplicative unity, it can be verified that

$$\mathbf{1} = \mathbf{e}_0 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.26}$$

satisfies both Eq. (2.9), because $\mathcal{M}(\mathbf{1}) \equiv \mathcal{I}$, and the existence condition, Eq. (2.11). □

Now, let us see how the parameters α and β feature the above-introduced bidimensional system of $\mathbb{H}$-numbers. The characteristic determinant of $\mathbf{z}$ is:

$$\det[\mathcal{M}(\mathbf{z})] = x^2 + \beta x y - \alpha y^2 . \tag{2.27}$$

Setting it equal to zero, gives a second order algebraic equation in the unknown x, whose solutions are

$$x = \frac{-\beta \pm \sqrt{\Delta}}{2}\, y, \tag{2.28}$$

where $\Delta = \beta^2 + 4\alpha$. The three main classes of bidimensional $\mathbb{H}$-numbers are distinguished according to the value of Δ [10, 11]. The first class is that of the *elliptic* $\mathbb{H}$-numbers, for which $\Delta < 0$; in this case, Eq. (2.28) gives no real solution, and therefore there are no divisors of zero. The second class is that of *hyperbolic* $\mathbb{H}$-numbers, for which $\Delta > 0$; in this case, the two solutions given by Eq. (2.28) represent two sets – two intersecting straight lines in the (x, y)-plane – of divisors of zero. The third class is that of *parabolic* $\mathbb{H}$-numbers, for which $\Delta = 0$; in this last case, the only set of divisors of zero is described by the equation $x = -\beta y/2$, which is a straight line (that becomes the y Cartesian axis for $\beta = 0$). The names of these three classes come from the geometrical figure of the conical curve described by Eq. (2.27) [11].

When $\beta = 0$ and $\alpha = \pm 1, 0$, the above three classes are further labelled as *canonical* [10, 11]. Standard complex numbers $\mathbb{Z}$, often indicated as $\mathbf{z} \equiv x + \mathbf{i}y \in \mathbb{Z}$, $\mathbf{i} = \sqrt{-1}$ being the imaginary unit, are the canonical elliptic $\mathbb{H}$-numbers ($\alpha = -1$ and $\beta = 0$). In this case, the usual imaginary unit, $\mathbf{i}$, is found to be

$$\mathbf{i} = \mathbf{e}_1 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.29}$$

This means that in our matrix representation of $\mathbf{z}$, see Eq. (2.23), when it is applied to standard complex numbers, x and y coincide with the usual real and imaginary parts of the number, respectively.

3. Properties of the Associated Matrix $\mathcal{M}$

The associated matrix $\mathcal{M}(\mathbf{x})$, that was defined in the previous Section, covers a fundamental role in the definition of $\mathbb{H}$-numbers and their operations. Let us explore the main properties of this kind of matrix, before going on with the study of $\mathbb{H}$-numbers. Some of these properties will allow us to introduce the functions of an $\mathbb{H}$-number.

3.1. Algebraic properties

Let us list the main algebraic properties of associated matrices. Some of them had been already introduced in the previous Section. The rather easy demonstrations are left to the reader.

$$\mathcal{M}(\mathbf{0}) = \mathcal{O}; \qquad \mathcal{M}(\mathbf{1}) = \mathcal{I}; \tag{3.1}$$

$$\mathcal{M}(a\mathbf{x} + b\mathbf{y}) = a\mathcal{M}(\mathbf{x}) + b\mathcal{M}(\mathbf{y}); \tag{3.2}$$

$$\mathcal{M}(\mathbf{x} \cdot \mathbf{y}) = \mathcal{M}(\mathbf{x})\mathcal{M}(\mathbf{y});$$

$$\mathcal{M}\left(\mathbf{x}^n\right) = \mathcal{M}^n(\mathbf{x}). \tag{3.3}$$

In Eq. (3.1) $\mathcal{O}$ and $\mathcal{I}$ are the all-zero $N \times N$ matrix and the identity $N \times N$ matrix, respectively; in Eq. (3.2) $a, b \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{H}_N$. In Eq. (3.3), n is an integer and when $n < 0$, $\mathbf{x}$ is assumed to be neither $\mathbf{0}$ nor a divisor of zero.

3.2. Spectral properties

Let us consider the eigenvalue problem for the matrix $\mathcal{M}(\mathbf{x})$, associated with the $\mathbb{H}$-number $\mathbf{x}$:

$$\mathcal{M}(\mathbf{x})\mathbf{u}_k(\mathbf{x}) = \lambda_k(\mathbf{x})\mathbf{u}_k(\mathbf{x})\,. \tag{3.4}$$

Here, $\mathbf{u}_k(\mathbf{x}) \in \mathbb{H}_N$. As for $\mathcal{M}(\mathbf{x})$, also for the scalar – real or complex – eigenvalues $\lambda_k(\mathbf{x})$ and eigenvectors $\mathbf{u}_k(\mathbf{x})$ the dependence on $\mathbf{x}$ should be rather meant as a dependence on the components of $\mathbf{x}$.

The eigenvectors $\mathbf{u}_k(\mathbf{x})$ are divisors of zero.
Proof – As a matter of fact, one can write Eq. (3.4) also as

$$[\mathbf{x} - \lambda_k(\mathbf{x})\mathbf{1}] \cdot \mathbf{u}_k(\mathbf{x}) = \mathbf{0}.$$

Because

$$\det\left[\mathcal{M}(\mathbf{x}) - \lambda_k(\mathbf{x})\mathcal{I}\right] = \det\left\{\mathcal{M}\left[\mathbf{x} - \lambda_k(\mathbf{x})\mathbf{1}\right]\right\} = 0$$

must hold true, $\mathbf{x} - \lambda_k(\mathbf{x})\mathbf{1}$ is a divisor of zero;[3] moreover, $\mathbf{u}_k(\mathbf{x}) \in \perp_{\mathbf{x}-\lambda_k(\mathbf{x})\mathbf{1}}$. Because all the elements of $\perp_{\mathbf{x}-\lambda_k(\mathbf{x})\mathbf{1}}$ are divisors of zero themselves, it is demonstrated that $\mathbf{u}_k(\mathbf{x})$ is a divisor of zero. □

If Eq. (3.4) admits N distinct eigenvalues, then the corresponding eigenvectors are mutually orthogonal, i.e.[4]

$$\mathbf{u}_k(\mathbf{x}) \cdot \mathbf{u}_n(\mathbf{x}) = \mathbf{0} \tag{3.5}$$

for $k \neq n$.
Proof – As a matter of fact, as Eq. (3.4) is true for $\mathbf{u}_k(\mathbf{x})$ and

$$\mathcal{M}(\mathbf{x})\mathbf{u}_n(\mathbf{x}) = \lambda_n(\mathbf{x})\mathbf{u}_n(\mathbf{x}) \tag{3.6}$$

is true for $\mathbf{u}_n(\mathbf{x})$, after right-multiplying Eq. (3.6) on both sides by $\mathbf{u}_k(\mathbf{x})$, one gets

$$\lambda_n(\mathbf{x})\mathbf{u}_n(\mathbf{x}) \cdot \mathbf{u}_k(\mathbf{x}) \equiv \mathcal{M}(\mathbf{x})\mathbf{u}_n(\mathbf{x}) \cdot \mathbf{u}_k(\mathbf{x}) \equiv \mathbf{x} \cdot \mathbf{u}_n(\mathbf{x}) \cdot \mathbf{u}_k(\mathbf{x})$$

---

[3] $\boldsymbol{x} - \lambda_k(\boldsymbol{x})\mathbf{1}$ is assumed to be different from $\mathbf{0}$.

[4] Hereafter, if not contrarily stated, we will assume that Eq. (3.4) admits N distinct eigenvalues.

$$= \mathbf{x} \cdot \mathbf{u}_k(\mathbf{x}) \cdot \mathbf{u}_n(\mathbf{x}) \equiv \mathcal{M}(\mathbf{x})\mathbf{u}_k(\mathbf{x}) \cdot \mathbf{u}_n(\mathbf{x}) \equiv \lambda_k(\mathbf{x})\mathbf{u}_k(\mathbf{x}) \cdot \mathbf{u}_n(\mathbf{x})\,.$$

From the above relationship, because the multiplication is commutative, it ensues

$$[\lambda_n(\mathbf{x}) - \lambda_k(\mathbf{x})]\,\mathbf{u}_n(\mathbf{x}) \cdot \mathbf{u}_k(\mathbf{x}) = \mathbf{0}\,.$$

Because the two eigenvalues are distinct by hypothesis, Eq. (3.5) is thus demonstrated. □

Given an $\mathbb{H}$-number $\mathbf{x}$, such that $\mathbf{x} \neq \mathbf{0}$, and its set of eigenvectors $\mathbf{u}_k(\mathbf{x})$, the following property holds true[5] if $\mathbf{x} \notin \perp_{\mathbf{u}_k(\mathbf{x})}$:

$$\mathbf{u}_k^2(\mathbf{x}) = \gamma_k(\mathbf{x})\mathbf{u}_k(\mathbf{x})\,, \tag{3.7}$$

where $\gamma_k(\mathbf{x})$ is a suitable scalar that depends on the components of $\mathbf{x}$.
Proof – As a matter of fact, $\mathbf{x}$ can be written as a linear combination of the eigenvectors of $\mathcal{M}(\mathbf{x})$:[6]

$$\mathbf{x} = \sum_{n=0}^{N-1} c_n(\mathbf{x})\mathbf{u}_n(\mathbf{x})\,, \tag{3.8}$$

where the $c_n(\mathbf{x})$ are scalar coefficients that depend on the components of $\mathbf{x}$. By substituting Eq. (3.8) into Eq. (3.4) and taking into account Eq. (3.5), one gets[7]

$$c_k(\mathbf{x})\mathbf{u}_k^2(\mathbf{x}) = \lambda_k(\mathbf{x})\mathbf{u}_k(\mathbf{x})\,. \tag{3.9}$$

Because we have assumed $\mathbf{x} \notin \perp_{\mathbf{u}_k(\mathbf{x})}$, it must be $c_k(\mathbf{x}) \neq 0$;[8] therefore, we can divide both sides of Eq. (3.9) by $c_k(\boldsymbol{x})$ to obtain Eq. (3.7) with

$$\gamma_k(\mathbf{x}) = \frac{\lambda_k(\mathbf{x})}{c_k(\mathbf{x})}. \quad \square$$

---

[5] Equation (3.7) can be alternatively written as

$$\mathcal{M}\,[\mathbf{u}_k(\mathbf{x})]\,\mathbf{u}_k(\mathbf{x}) = \gamma_k(\mathbf{x})\mathbf{u}_k(\mathbf{x})\,,$$

that means that $\mathbf{u}_k(\mathbf{x})$ is eigenvector of the matrix associated with itself.

[6] As known from linear algebra, if Eq. (3.4) admits N distinct eigenvalues, then the corresponding N eigenvectors are linearly independent. In such a case, any column vector (i.e. $\mathbb{H}$-number) can be written as a linear combination of the eigenvectors.

[7] We recall that $\mathcal{M}(\mathbf{x})\mathbf{u}_k(\mathbf{x}) = \mathbf{x} \cdot \mathbf{u}_k(\mathbf{x})$.

[8] If $c_k(\mathbf{x})$ were zero, one would get $\mathbf{x} \cdot \mathbf{u}_k(\mathbf{x}) = \mathbf{0}$ from Eq. (3.8), and therefore $\mathbf{x} \in \perp_{\mathbf{x}_k(\mathbf{x})}$ would be true, in contrast with our hypothesis.

The eigenvectors of the associated matrix $\mathcal{M}(\mathbf{x})$ do not depend on the components of $\mathbf{x}$.[9]

Proof – As a matter of fact, because the eigenvectors $\mathbf{u}_k(\mathbf{x})$ form a linearly independent system, given any $\mathbf{y} \in \mathbb{H}_N$ one can write it as a linear combination of the eigenvectors:

$$\mathbf{y} = \sum_{n=0}^{N-1} c_n(\mathbf{x}, \mathbf{y})\mathbf{u}_n(\mathbf{x}) \,, \tag{3.10}$$

where the real coefficients $c_n(\mathbf{x}, \mathbf{y})$ generally depend on the components of $\mathbf{x}$ and $\mathbf{y}$. Therefore, thanks to Eqs. (3.10), (3.2), (3.5), and (3.7), one can write

$$\begin{aligned}\mathcal{M}(\mathbf{y})\mathbf{u}_k(\mathbf{x}) &= \sum_{n=0}^{N-1} c_n(\mathbf{x}, \mathbf{y})\mathcal{M}\left[\mathbf{u}_n(\mathbf{x})\right]\mathbf{u}_k(\mathbf{x}) = \sum_{n=0}^{N-1} c_n(\mathbf{x}, \mathbf{y})\mathbf{u}_n(\mathbf{x}) \cdot \mathbf{u}_k(\mathbf{x}) \\ &= c_k(\mathbf{x}, \mathbf{y})\mathbf{u}_k^2(\mathbf{x}) = c_k(\mathbf{x}, \mathbf{y})\gamma_k(\mathbf{x})\mathbf{u}_k(\mathbf{x}) \,. \end{aligned} \tag{3.11}$$

Equation (3.11) shows that $\mathbf{u}_k(\mathbf{x})$ is eigenvector of $\mathcal{M}(\mathbf{y})$ as well, whatever $\mathbf{y}$ is. □

This result demonstrates that the eigenvectors form a set that only depends on the system of $\mathbb{H}$-numbers under consideration. For this reason, hereafter we will drop the dependence on $\mathbf{x}$ from the eigenvectors.

To build *normalized* eigenvectors, let us set[10]

$$\mathbf{v}_k = \frac{1}{\gamma_k}\mathbf{u}_k \,. \tag{3.12}$$

It follows from Eq. (3.5):

$$\mathbf{v}_k \cdot \mathbf{v}_n = \mathbf{0} \tag{3.13}$$

for $k \neq n$. Moreover, from Eq. (3.7) one gets

$$\mathbf{v}_k^2 = \mathbf{v}_k \,. \tag{3.14}$$

By virtue of Eqs. (3.13) and (3.14), we shall say that the eigenvectors $\mathbf{v}_k$ form a set of *orthonormal* $\mathbb{H}$-numbers.

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[9] This implies that Eq. (3.7) is true in the following $\mathbf{x}$-independent form:

$$\mathbf{u}_k^2 = \gamma_k \mathbf{u}_k.$$

[10] Note that the original and normalized eigenvectors, $\mathbf{u}_k$ and $\mathbf{v}_k$, share the same eigenvalues $\lambda_k(\mathbf{x})$ for a chosen $\mathbb{H}$-number $\mathbf{x}$.

These orthonormal eigenvectors $\mathbf{v}_k$ are a complete set [12], therefore any $\mathbb{H}$-number $\mathbf{x}$ can be expressed as a linear superposition of them.
Proof – As a matter of fact, knowing that

$$\mathcal{M}(\mathbf{x})\mathbf{v}_k \equiv \mathbf{x} \cdot \mathbf{v}_k = \lambda_k(\mathbf{x})\mathbf{v}_k\,, \tag{3.15}$$

one can verify – by recalling completeness of the eigenvectors of a matrix – that

$$\mathbf{x} = \sum_{k=0}^{N-1} \lambda_k(\mathbf{x})\mathbf{v}_k\,. \quad \Box \tag{3.16}$$

Equation (3.16), together with Eqs. (3.13) and (3.14), plays a fundamental role for the system of $\mathbb{H}$-numbers: it states that the eigenvectors $\mathbf{v}_k$ can be used as a new set of unit vectors (i.e. a basis) instead of the natural unit vectors $\mathbf{e}_k$, and that the components of any $\mathbb{H}$-number for them are the eigenvalues of the associated matrix.[11] Equation (3.16) will come handy when functions of $\mathbb{H}$-numbers will be dealt with.

As a consequence of Eq. (3.16), let us note that the unity $\mathbf{1}$ can be written as

$$\mathbf{1} = \sum_{k=0}^{N-1} \mathbf{v}_k\,. \tag{3.17}$$

Proof – As a matter of fact, being $\mathcal{M}(\mathbf{1}) = \mathcal{I}$, the eigenvalues of the associated matrix are all equal to 1, so that Eq. (3.17) follows. □

3.3. Properties of the eigenvalues

The following properties of the eigenvalues of the associated matrix can be straightforwardly demonstrated to hold true:

$$\lambda_k(\mathbf{0}) = 0\,; \qquad \lambda_k(\mathbf{1}) = 1\,; \tag{3.18}$$

$$\lambda_k(a\mathbf{x} + b\mathbf{y}) = a\lambda_k(\mathbf{x}) + b\lambda_k(\mathbf{y})\,; \tag{3.19}$$

$$\lambda_k(\mathbf{x} \cdot \mathbf{y}) = \lambda_k(\mathbf{x})\lambda_k(\mathbf{y})\,; \tag{3.20}$$

$$\lambda_k\left(\mathbf{x}^n\right) = \lambda_k^n(\mathbf{x})\,. \tag{3.21}$$

In the above equations, $a, b \in \mathbb{R}$, $\mathbf{x}$ and $\mathbf{y}$ are two $\mathbb{H}$-numbers and n is an integer. When $n < 0$ in Eq. (3.21), $\mathbf{x}$ is assumed to be neither zero nor a divisor of zero.

[11] Generally, the components $\lambda_k(\mathbf{x})$ of $\mathbf{x}$ can be complex in this new $\{\mathbf{v}_k\}$ basis. This fact, however, does not constitute a problem for the developments that will follow.

3.4. More about divisors of zero

If $\mathbf{x} \in \mathbb{H}_N$ is a divisor of zero, then

$$\mathbf{x} \notin \perp_{\mathbf{x}} \ . \tag{3.22}$$

Proof − As a matter of fact, if $\mathbf{x} \in \perp_{\mathbf{x}}$ were true, one would get $\mathbf{x}^2 = \mathbf{0}$. On the other hand, Eq. (3.21) allows writing

$$\mathbf{x}^2 = \sum_{k=0}^{N-1} \lambda_k^2(\mathbf{x}) \mathbf{v}_k \, . \tag{3.23}$$

Because $\mathbf{x} \neq \mathbf{0}$ for its being a divisor of zero, at least one eigenvalue $\lambda_k(\mathbf{x})$ must be non-zero, so that also $\mathbf{x}^2 \neq \mathbf{0}$. This demonstrates Eq. (3.22) to be true.[12] □

3.5. Modulus of a hypercomplex number

Let us consider $\mathbf{x} \in \mathbb{H}_N$ for which $\det[\mathcal{M}(\mathbf{x})] \geq 0$. The *modulus* of $\mathbf{x}$, indicated as $|\mathbf{x}|$, is defined as

$$|\mathbf{x}| = \{\det[\mathcal{M}(\mathbf{x})]\}^{1/N} \ . \tag{3.24}$$

Demonstrating that the modulus of a divisor of zero is zero is straightforward.

3.6. Conjugations of a hypercomplex number

Let us consider $\mathbf{x} \in \mathbb{H}_N$. We define $N-1$ conjugations of $\mathbf{x}$, indicated with ${}^1\overline{\mathbf{x}}, {}^2\overline{\mathbf{x}}, \ldots, {}^{N-1}\overline{\mathbf{x}}$, as

$${}^s\overline{\mathbf{x}} = \sum_{k=0}^{N-1} \lambda_{[k+s]}(\mathbf{x}) \mathbf{v}_k \qquad (s = 1, 2, \ldots, N-1) \, , \tag{3.25}$$

where $[k+s]$ is the congruent of $k+s$ module N.[13] Equation (3.25) states that the s^{th} conjugation of $\mathbf{x}$ is obtained by scaling in a cyclic way the subscript of

[12] Therefore, the only solution to equation

$$\mathbf{x}^2 = \mathbf{0}$$

is $\mathbf{x} = \mathbf{0}$.

[13] We recall that, given a positive integer q, the congruent $[q]$ of q module N is equal to $q-mN$, where m is a suitable integer number such that $0 \leq q-mN \leq N-1$.

the eigenvalues by s places with respect to those of the eigenvectors $\mathbf{v}_k$ of $\mathbf{x}$. One can verify that the conjugations satisfy the condition

$$\mathbf{x} \cdot {}^1\overline{\mathbf{x}} \cdot {}^2\overline{\mathbf{x}} \cdots {}^{N-1}\overline{\mathbf{x}} = \det[\mathcal{M}(\mathbf{x})]\mathbf{1}\,. \tag{3.26}$$

To demonstrate it, one has to consider that the product of the eigenvalues of a matrix is equal to the matrix determinant.

As one can demonstrate, the hypercomplex conjugations satisfy the following properties:

$$^s\overline{\mathbf{0}} = \mathbf{0}\,; \qquad {}^s\overline{\mathbf{1}} = \mathbf{1}\,; \tag{3.27}$$

$$^r\overline{({}^s\overline{\mathbf{x}})} = {}^{[r+s]}\overline{\mathbf{x}}\,; \tag{3.28}$$

$$^s\overline{(\mathbf{x}\cdot\mathbf{y})} = {}^s\overline{\mathbf{x}} \cdot {}^s\overline{\mathbf{y}}\,; \tag{3.29}$$

$$^s\overline{(a\mathbf{x} + b\mathbf{y})} = a\,{}^s\overline{\mathbf{x}} + b\,{}^s\overline{\mathbf{y}}\,; \tag{3.30}$$

$$^s\overline{(\mathbf{x}^n)} = ({}^s\overline{\mathbf{x}})^n\;. \tag{3.31}$$

In the above equations, $a, b \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{H}_N$, and n is an integer. In Eq. (3.31), $\det[\mathcal{M}(\boldsymbol{x})] \neq 0$ is assumed when $n < 0$.

Because ${}^s\overline{\mathbf{x}}$ is an $\mathbb{H}$-number, it can be written with an expression similar to Eq. (3.16), that is

$$^s\overline{\mathbf{x}} = \sum_{k=0}^{N-1} \lambda_k({}^s\overline{\mathbf{x}})\mathbf{v}_k\,. \tag{3.32}$$

By comparing Eqs. (3.25) and (3.32), one finds that

$$\lambda_k({}^s\overline{\mathbf{x}}) = \lambda_{[k+s]}(\mathbf{x})\,. \tag{3.33}$$

4. Functions of Hypercomplex Variable

4.1. Analytic continuation

Let us consider a real function of real variable, $f : \mathbb{R} \to \mathbb{R}$. Let us assume that f can be written as a power series in its domain, that is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n\,, \tag{4.1}$$

where $a_n \in \mathbb{R}$ are suitable coefficients. Similarly to the case of complex numbers, the *analytic continuation* $\mathbf{f}$ of f is defined as

$$\mathbf{f}(\mathbf{x}) = \sum_{n=0}^{\infty} a_n \mathbf{x}^n\,. \tag{4.2}$$

Clearly, $\mathbf{f} : \mathbb{H}_N \to \mathbb{H}_N$.

We are going to see that it is possible to establish a link between the functions defined by means of Eq. (4.2) and matrix functions.

For what reported in Sec. 2.3.6, Eq. (4.2) can be written also as

$$\mathbf{f}(\mathbf{x}) = \sum_{n=0}^{\infty} a_n \mathcal{M}^n(\mathbf{x})\mathbf{1} = \left[\sum_{n=0}^{\infty} a_n \mathcal{M}^n(\mathbf{x})\right]\mathbf{1}\,. \tag{4.3}$$

Note that the series within square brackets, is the power series of a matrix function [12] that will be indicated with $\mathcal{F}[\mathcal{M}(\mathbf{x})]$. Therefore, if this series converges, also the series in Eq. (4.2) converges. Moreover, the following fundamental relationship can be stated between the analytic continuation $\mathbf{f}$ of f and the matrix function $\mathcal{F}$ associated with f:

$$\boldsymbol{f}(\boldsymbol{x}) = \mathcal{F}[\mathcal{M}(\boldsymbol{x})]\,\mathbf{1}. \tag{4.4}$$

As previously anticipated, Eq. (4.4) enables us to study the properties of hypercomplex functions by exploiting the results known for matrix functions.

A first important result is the following: the theory of matrix functions states that if $\mathcal{M}(\mathbf{x})$ admits N normalized eigenvectors $\mathbf{v}_k$ that form a basis [12], then

$$\mathcal{F}[\mathcal{M}(\mathbf{x})]\,\mathbf{v}_k = f[\lambda_k(\mathbf{x})]\,\mathbf{v}_k\,. \tag{4.5}$$

Therefore, by virtue of Eq. (3.17) we can write the unity $\mathbf{1}$ as a sum of the $\mathbf{v}_k$, and Eq. (4.4) becomes:

$$\mathbf{f}(\mathbf{x}) = \mathcal{F}[\mathcal{M}(\mathbf{x})]\sum_{k=0}^{N-1}\mathbf{v}_k = \sum_{k=0}^{N-1}\mathcal{F}[\mathcal{M}(\mathbf{x})]\,\mathbf{v}_k = \sum_{k=0}^{N-1} f[\lambda_k(\mathbf{x})]\,\mathbf{v}_k\,. \tag{4.6}$$

Equation (4.6) establishes a very important link between f and its analytic continuation $\mathbf{f}$. Generally, it holds true even when the eigenvalues of $\mathcal{M}(\mathbf{x})$ are complex numbers. In such a case, one has to consider, in the right-side member of Eq. (4.6), the *complex* analytic continuation of f. The resulting hypercomplex function will have, nonetheless, pure real components in the $\{\mathbf{e}_k\}$ basis, because Eq. (4.3) contains just real coefficients and matrices with real entries.

4.1.1. Example: functions of a bidimensional hypercomplex variable

As one can verify, for bidimensional $\mathbb{H}$-numbers the eigenvalues of the associated matrix are

$$\lambda_{\pm}(x, y) = x + \frac{1}{2}\left(\beta \pm \sqrt{\Delta}\right) y\,. \tag{4.7}$$

To them, the following orthonormal eigenvectors correspond

$$\mathbf{v}_{\pm} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} \frac{\sqrt{\Delta}\mp\beta}{2} \\ \pm 1 \end{pmatrix} , \tag{4.8}$$

where we have assumed, for the moment, $\Delta \neq 0$ (*elliptic* and *hyperbolic* numbers [11]).[14] By applying Eq. (4.6), one gets:

$$\mathbf{f}\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} f_0(x,y) \\ f_1(x,y) \end{pmatrix} , \tag{4.9}$$

with

$$\begin{aligned} f_0(x,y) &= \frac{1}{2}\left[f\left(x+\frac{\beta}{2}y+\frac{\sqrt{\Delta}}{2}y\right)+f\left(x+\frac{\beta}{2}y-\frac{\sqrt{\Delta}}{2}y\right)\right] \\ &- \frac{\beta}{2\sqrt{\Delta}}\left[f\left(x+\frac{\beta}{2}y+\frac{\sqrt{\Delta}}{2}y\right)-f\left(x+\frac{\beta}{2}y-\frac{\sqrt{\Delta}}{2}y\right)\right], \end{aligned} \tag{4.10}$$

and

$$f_1(x,y) = \frac{1}{\sqrt{\Delta}}\left[f\left(x+\frac{\beta}{2}y+\frac{\sqrt{\Delta}}{2}y\right)-f\left(x+\frac{\beta}{2}y-\frac{\sqrt{\Delta}}{2}y\right)\right]. \tag{4.11}$$

In the limit $\Delta \to 0$, the above equations give the following expression for *parabolic* numbers [11, 13]:

$$\mathbf{f}\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\Delta\to 0} = \begin{pmatrix} f\left(x+\frac{\beta}{2}y\right) - \frac{\beta}{2}yf'\left(x+\frac{\beta}{2}y\right) \\ yf'\left(x+\frac{\beta}{2}y\right) \end{pmatrix} . \tag{4.12}$$

In Ref. [13], it has been shown that the definition of functions of a parabolic variable allows demonstrating, by means of plain algebra, the main theorems concerning the derivative of functions of a real variable.

A specialization of Eqs. (4.10) and (4.11) to complex numbers ($\alpha = -1$, $\beta = 0$, $\Delta = 2\mathbf{i}$) leads to well known results for the real and imaginary parts, f_0 and f_1, of $\mathbf{f}$, which are

$$f_0(x,y) = \frac{1}{2}\left[f(x+\mathbf{i}y)+f(x-\mathbf{i}y)\right] , \qquad f_1(x,y) = \frac{1}{2\mathbf{i}}\left[f(x+\mathbf{i}y)-f(x-\mathbf{i}y)\right] . \tag{4.13}$$

[14] The case of *parabolic* numbers [11, 13], for which $\Delta = 0$, will be tackled later by applying $\Delta \to 0$ to the results obtained with $\Delta \neq 0$.

4.2. Properties of hypercomplex functions

We list below, leaving the straightforward demonstrations to the reader, some properties of the functions of hypercomplex variable:

$$\text{if} \quad f(ax+by) = af(x)+bf(y) \quad \text{then} \quad \mathbf{f}(a\mathbf{x}+b\mathbf{y}) = a\mathbf{f}(\mathbf{x})+b\mathbf{f}(\mathbf{y})\,; \tag{4.14}$$

$$\text{if} \quad f(xy) = f(x) + f(y) \quad \text{then} \quad \mathbf{f}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{y})\,; \tag{4.15}$$

$$\text{if} \quad f(x+y) = f(x)f(y) \quad \text{then} \quad \mathbf{f}(\mathbf{x}+\mathbf{y}) = \mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{y})\,. \tag{4.16}$$

In the above equations, $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{H}_N$.

Other interesting properties can be investigated by using Eq. (4.6). By means of it, for instance, one can define the analytic continuations of the exponential and logarithmic functions. Coherently with the adopted notation, we will indicate these analytic continuations in bold letters, that is **exp** and **ln**. Regarding these two functions, from Eqs. (4.15) and (4.16) it follows that they possess the same properties of the corresponding real and complex functions:

$$\mathbf{ln}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{ln}(\mathbf{x}) + \mathbf{ln}(\mathbf{y});$$

$$\mathbf{exp}(\mathbf{x}+\mathbf{y}) = \mathbf{exp}(\mathbf{x})\cdot\mathbf{exp}(\mathbf{y}).$$

5. Derivatives of a Hypercomplex Function

Before introducing the derivative, it is useful to introduce the limit of a function of hypercomplex variable. Given $\mathbf{x} \in \mathbb{H}_N$ and $\mathbf{f} : \mathbb{H}_N \to \mathbb{H}_N$, we define the limit of $\mathbf{f}(\mathbf{x})$ for $\mathbf{x} \to \mathbf{y}$ to be equal to $\boldsymbol{\ell} \equiv (\ell_0, \ell_1, \ldots, \ell_{N-1})$ if

$$\lim_{x_0 \to y_0} \lim_{x_1 \to y_1} \ldots \lim_{x_{N-1} \to y_{N-1}} f_k(\mathbf{x}) = \ell_k \text{ for } k = 0, 1, \ldots, N-1, \tag{5.1}$$

under the assumption that all the limits of the components do exist.

5.1. Derivative with respect to the hypercomplex variable

Assuming that $f(x)$ is a derivable function, let us define the derivative of $\mathbf{f}(\mathbf{x})$ with respect to the hypercomplex variable $\mathbf{x}$. The *incremental ratio* of this function is defined as

$$\frac{\Delta\mathbf{f}}{\Delta\mathbf{x}}(\mathbf{x}, \mathbf{t}) = \frac{\mathbf{f}(\mathbf{x}+\mathbf{t}) - \mathbf{f}(\mathbf{x})}{\mathbf{t}}\,, \tag{5.2}$$

with $\mathbf{t} \in \mathbb{H}_\mathbf{N}$. Obviously, the division in the previous definition is defined only if $\det[\mathcal{M}(\mathbf{t})] \neq \mathbf{0}$. Under such an assumption, we define the *derivative* of $\mathbf{f}(\mathbf{x})$ with respect to $\mathbf{x}$ as the limit of the incremental ratio for $\mathbf{t} \to \mathbf{0}$:

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) = \lim_{\mathbf{t}\to\mathbf{0}} \frac{\Delta\mathbf{f}}{\Delta\mathbf{x}}(\mathbf{x}, \mathbf{t}) \,, \tag{5.3}$$

and we will say that such a derivative exists if the above limit exists and is independent on the order chosen for evaluating the limits along the components of $\mathbf{t}$, see Eq. (5.1). Moreover, the derivative cannot be defined in $\mathbf{x}$ if $\mathbf{x}$ is a divisor of zero. Alternatively, we could introduce the derivative by applying Eq. (4.6) to the real function $f'(x)$ – the derivative of $f(x)$ – that is

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) = \sum_{k=0}^{N-1} f'[\lambda_k(\mathbf{x})]\mathbf{v}_k \,. \tag{5.4}$$

The two above definitions, Eqs. (5.3) and (5.4), are completely equivalent if $\mathbf{t}$ is not a divisor of zero.
Proof – As a matter of fact, Eq. (5.3), together with Eq. (4.6), gives:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) &= \lim_{\mathbf{t}\to\mathbf{0}} \sum_{k=0}^{N-1} \{f[\lambda_k(\mathbf{x}+\mathbf{t})] - \mathbf{f}[\lambda_\mathbf{k}(\mathbf{x})]\}\, \mathcal{M}^{-1}(\mathbf{t})\mathbf{v}_\mathbf{k} \\ &= \lim_{\mathbf{t}\to\mathbf{0}} \sum_{k=0}^{N-1} \frac{f[\lambda_k(\mathbf{x}) + \lambda_k(\mathbf{t})] - \mathbf{f}[\lambda_\mathbf{k}(\mathbf{x})]}{\lambda_k(\mathbf{t})}\mathbf{v}_k = \sum_{k=0}^{N-1} f'[\lambda_k(\mathbf{x})]\mathbf{v}_k \,, \end{aligned}$$

because $\lim_{\mathbf{t}\to\mathbf{0}} \lambda_k(\mathbf{t}) = \mathbf{0}$. □

Equation (5.4) does not need the introduction of an increment $\mathbf{t}$ for the definition of the derivative of $\mathbf{f}(\mathbf{x})$, and therefore it is preferable with respect to the definition given in Eq. (5.2).

As far as the higher-order derivatives are concerned, one can demonstrate that a proper definition is

$$\frac{\mathrm{d}^n\mathbf{f}}{\mathrm{d}\mathbf{x}^n}(\mathbf{x}) = \sum_{k=0}^{N-1} f^{(n)}[\lambda_k(\mathbf{x})]\mathbf{v}_k \,,$$

where n is any positive integer.

With the definition given in Eq. (5.4), one can demonstrate that the common properties of the derivative of real functions – derivative of the sum of two functions, of the product of two functions, of the ratio of two function, of the function of a function, etc. – can be straightforwardly extended to the derivative of functions of hypercomplex variable.

5.2. Partial derivatives

Equation (4.6) can help also in introducing the partial derivative of $\mathbf{f}(\mathbf{x})$ with respect to one component of $\mathbf{x}$, say x_n. The resulting definition is

$$\frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}) = \frac{\partial}{\partial x_n} \sum_{k=0}^{N-1} f[\lambda_k(\mathbf{x})]\mathbf{v}_k = \sum_{k=0}^{N-1} f'[\lambda_k(\mathbf{x})] \frac{\partial \lambda_k}{\partial x_n}(\mathbf{x})\mathbf{v}_k \,. \tag{5.5}$$

Thanks to Eq. (3.20), according to which the product of two $\mathbb{H}$-numbers can be evaluated, in the orthonormal basis $\{\mathbf{v}_k\}$, by multiplying their components one by one (internal product), the above equation gives

$$\frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) \cdot \frac{\partial}{\partial x_n} \sum_{k=0}^{N-1} \lambda_k(\mathbf{x})\mathbf{v}_k = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) \cdot \frac{\partial \mathbf{x}}{\partial x_n} \,. \tag{5.6}$$

Noticing that $\partial\mathbf{x}/\partial x_n \equiv \mathbf{e}_n$, one gets the following fundamental result, which relates the derivative with the partial derivatives of a function of hypercomplex variable:

$$\frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) \cdot \mathbf{e}_n \,. \tag{5.7}$$

In a similar way, one can verify that

$$\frac{\partial^2 \mathbf{f}}{\partial x_m \partial x_n}(\mathbf{x}) = \frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\mathbf{x}^2} \cdot \mathbf{e}_m \cdot \mathbf{e}_n$$

holds true, and that analogous relationships hold true for higher-order derivatives.

From Eq. (5.7), the following relationship among the partial derivatives of $\mathbf{f}(\mathbf{x})$ follows:

$$\mathbf{e}_m \cdot \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}) = \mathbf{e}_n \cdot \frac{\partial \mathbf{f}}{\partial x_m}(\mathbf{x}) \tag{5.8}$$

and is true for any $m, n = 0, 1, \ldots, N-1$. Equation (5.8) can be regarded as the vectorial form of the generalized Cauchy-Riemann (GCR) conditions [3].

A function of hypercomplex variable that satisfies the GCR conditions – i.e. Eq. (5.8) – will be named *holomorphic* (or holomorphic analytic). Clearly, any analytic continuation of a real function is holomorphic, because Eq. (5.8) has been derived for functions which are analytic continuations.

5.3. Components of the derivative operator

One can define a *derivative operator*, $\mathrm{d}/\mathrm{d}\mathbf{x}$, by *formally* right-multiplying it by the function subjected to the derivative operation:

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \cdot \mathbf{f}(\mathbf{x}) \equiv \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x})\,. \tag{5.9}$$

In such a case, by formally dealing with the derivative operator as if it were an $\mathbb{H}$-number, one can decompose it in the orthonormal basis $\{\mathbf{v}_k\}$:

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \sum_{k=0}^{N-1} \lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) \mathbf{v}_k\,. \tag{5.10}$$

Because

$$\mathbf{f}(\mathbf{x}) = \sum_{k=0}^{N-1} f\left[\lambda_k(\mathbf{x})\right] \mathbf{v}_k\,, \tag{5.11}$$

by formally multiplying Eq. (5.10) by Eq. (5.11) and recalling that the $\mathbf{v}_k$ are orthonormal, one gets:

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \cdot \mathbf{f}(\mathbf{x}) \equiv \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) = \sum_{k=0}^{N-1} \lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) f\left[\lambda_k(\mathbf{x})\right] \mathbf{v}_k\,. \tag{5.12}$$

By comparing Eq. (5.12) with Eq. (5.4), one gets the following expression for the eigenvalues of the derivative operator:

$$\lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda_\mathrm{k}(\mathbf{x})}\,. \tag{5.13}$$

Thus, these eigenvalues are the derivative operators applied to the single components of $\mathbf{f}(\mathbf{x})$ in the $\{\mathbf{v}_k\}$ basis.

5.4. Derivative with respect to the conjugated variables

By exploiting the above results, one can also decompose, within the $\{\mathbf{v}_k\}$ basis, the derivative operator with respect to ${}^s\overline{\mathbf{x}}$, which is the s-conjugated of $\mathbf{x}$. Equation (5.13) formally gives

$$\lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda_\mathrm{k}({}^s\overline{\mathbf{x}})}\,. \tag{5.14}$$

Because Eq. (3.33) holds true, Eq. (5.14) can also be written as

$$\lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda_{[\mathrm{k+s}]}(\mathbf{x})}\,, \tag{5.15}$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} = \sum_{k=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\lambda_{[\mathrm{k+s}]}(\mathbf{x})} \mathbf{v}_k\,. \tag{5.16}$$

Moreover, the latter two expressions give

$$\lambda_k \left(\frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} \right) = \lambda_{[k+s]} \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \right)\,. \tag{5.17}$$

By using the above results, it should be noticed that, if $\mathbf{f}(\mathbf{x})$ is a holomorphic function, one gets for $s \neq 0$:

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}}(\mathbf{x}) \equiv \frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} \cdot \mathbf{f}(\mathbf{x}) \equiv \sum_{k=0}^{N-1} \frac{\mathrm{d}f[\lambda_{\mathrm{k}}(\mathbf{x})]}{\mathrm{d}\lambda_{[\mathrm{k+s}]}(\mathbf{x})} \mathbf{v}_k = \mathbf{0}\,. \tag{5.18}$$

This last equation states that the derivatives of a holomorphic function $\mathbf{f}(\mathbf{x})$ with respect to the conjugated variables ${}^s\overline{\mathbf{x}}$ $(s \neq 0)$ are null. Such a property is a generalization of what happens in the system of complex numbers for complex holomorphic functions.

5.5. Characteristic differential equation

Equation (5.18) gives rise to systems of differential equations of order lower than or equal to N. The components of any holomorphic function must satisfy such differential equations. For instance, let us consider the composite operator $(\mathrm{d}/\mathrm{d}\mathbf{x}) \cdot (\mathrm{d}/\mathrm{d}\,{}^s\overline{\mathbf{x}})$, with $s \neq 0$. Because

$$\frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{0}\,,$$

also

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}\,{}^s\overline{\mathbf{x}}} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{5.19}$$

holds true. By setting equal to zero the N components of the left-hand side of Eq. (5.19), one will find a system of differential equations of order 2 for the N components of $\mathbf{f}(\mathbf{x})$.

An interesting result comes from the application of the operator

$$\prod_{s=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\,{}^{s}\overline{\mathbf{x}}} \equiv \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}\,{}^{1}\overline{\mathbf{x}}} \cdots \frac{\mathrm{d}}{\mathrm{d}\,{}^{N-1}\overline{\mathbf{x}}}$$

to a holomorphic function $\mathbf{f}(\mathbf{x})$. Indeed, by recalling orthonormality of the unit vectors $\mathbf{v}_k$ and Eq. (5.16), one gets

$$\begin{aligned}\left(\prod_{s=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\,{}^{s}\overline{\mathbf{x}}}\right) \cdot \mathbf{f}(\mathbf{x}) &= \left[\prod_{s=0}^{N-1} \left(\sum_{k=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\lambda_{[\mathrm{k+s}]}(\mathbf{x})} \mathbf{v}_k\right)\right] \cdot \mathbf{f}(\mathbf{x}) \\ &= \left(\prod_{s=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\lambda_{\mathrm{s}}(\mathbf{x})}\right) \cdot \left(\sum_{k=0}^{N-1} \mathbf{v}_k\right) \cdot \mathbf{f}(\mathbf{x}) = \left[\prod_{s=0}^{N-1} \lambda_s \left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right)\right] \cdot \mathbf{1} \cdot \mathbf{f}(\mathbf{x}) \\ &= \left|\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right|^N \cdot \mathbf{f}(\mathbf{x}) = \mathbf{0}\,. \end{aligned} \tag{5.20}$$

By setting as equal to zero the N components of Eq. (5.20), one finds that *all* the components of $\mathbf{f}(\mathbf{x})$ must satisfy *the same* differential equation of order N, that we will name as the *characteristic equation* of the system of $\mathbb{H}$-numbers. Note that the coefficients of such an equation are real numbers.

A more explicit form of the characteristic equation can be obtained by evaluating the N^{th} power of the symbolic modulus of the derivative operator that appears in the last row of Eq. (5.20). Indeed, after recalling Eqs. (3.24) and (5.13) and the known property that the product of the eigenvalues of a matrix is equal to the matrix determinant, one has

$$\left|\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right|^N = \prod_{s=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}\lambda_{\mathrm{s}}(\mathbf{x})} = \prod_{s=0}^{N-1} \left[\sum_{k=0}^{N-1} \frac{\partial x_k}{\partial \lambda_s(\mathbf{x})} \frac{\partial}{\partial x_k}\right]. \tag{5.21}$$

Because

$$\mathbf{x} = \sum_{s=0}^{N-1} \lambda_s(\mathbf{x}) \mathbf{v}_s = \sum_{k=0}^{N-1} \left[\sum_{s=0}^{N-1} \lambda_s(\mathbf{x}) v_{sk}\right] \mathbf{e}_k\,,$$

where the v_{sk} are the components of $\mathbf{v}_s$ in the $\{\mathbf{e}_k\}$ basis,

$$\mathbf{v}_s = \sum_{k=0}^{N-1} v_{sk} \mathbf{e}_k\,,$$

it follows that the k^{th} component of $\mathbf{x}$ is

$$x_k = \sum_{s=0}^{N-1} \lambda_s(\boldsymbol{x}) v_{sk},$$

from which one finds

$$\frac{\partial x_k}{\partial \lambda_s(\mathbf{x})} = v_{sk}\,. \tag{5.22}$$

By substituting Eq. (5.22) into Eq. (5.21) and recalling Eq. (5.20), the following explicit form of the characteristic equation follows:

$$\prod_{s=0}^{N-1} \left(\sum_{k=0}^{N-1} v_{sk} \frac{\partial}{\partial x_k} \right) \mathbf{f}(\mathbf{x}) = \mathbf{0}\,. \tag{5.23}$$

Since in Eq. (5.23) the operator applied to $\mathbf{f}(\mathbf{x})$ is a real one, Eq. (5.23) must hold true for all the components – in the $\{\mathbf{e}_k\}$ basis – of $\mathbf{f}(\mathbf{x})$, that is

$$\prod_{s=0}^{N-1} \left(\sum_{k=0}^{N-1} v_{sk} \frac{\partial}{\partial x_k} \right) f_k(x_0, x_1, \ldots, x_{N-1}) = 0 \tag{5.24}$$

for any $k = 0, 1, \ldots, N-1$.

5.5.1. Example: characteristic equation for bidimensional hypercomplex numbers

After recalling the form of the orthonormal unit vectors for bidimensional $\mathbb{H}$-numbers, see Eq. (4.8), the characteristic equation resulting from Eq. (5.23) is

$$\left(\frac{\sqrt{\Delta} - \beta}{2\sqrt{\Delta}} \frac{\partial}{\partial x} + \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial y} \right) \left(\frac{\sqrt{\Delta} + \beta}{2\sqrt{\Delta}} \frac{\partial}{\partial x} - \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial y} \right) \begin{pmatrix} f_0(x,y) \\ f_1(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A few mathematical manipulations transform this equation into a more compact form, that is

$$\left(\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial^2}{\partial x\, \partial y} - \frac{\partial^2}{\partial y^2} \right) \begin{pmatrix} f_0(x,y) \\ f_1(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.25}$$

Equation (5.25) is a partial differential equation of elliptic, parabolic or hyperbolic type [14], depending on the values of α and β. In particular, Eq. (5.25) becomes the well known Laplace equation for *canonical elliptic* bidimensional $\mathbb{H}$-numbers ($\alpha = -1$, $\beta = 0$) – which coincide with complex numbers – and the wave equation for *canonical hyperbolic* bidimensional $\mathbb{H}$-numbers ($\alpha = 1$, $\beta = 0$).

6. Conclusions

The theory of commutative hypercomplex numbers whith unity has been exposed by means of the linear algebra of matrices. In spite of the use of an algebraic method, we have formalized the theory of holomorphic functions and differential calculus. Thanks to these important features and the commutative property, we can assume these systems of numbers to be extensions of real and complex numbers.

Moreover, the fact that within bidimensional systems the holomorphic functions satisfy partial differential equations of physical relevance, stimulates the extension of our investigation to multidimensional systems.

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