REPORT

THE EXCEPTIONAL LIE GROUP E_8

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Chapter 1 Clifford Algebras

Throughout this project, we let V be a finite dimensional real vector space, together with a symmetric bilinear form $B : V \times V \longrightarrow \mathbb{R}$ and we let $Q : V \longrightarrow \mathbb{R}$ be the associated quadratic form, $Q(v) = B(v, v)$ for $v \in V$.

We recall from the class that if $V = \mathbb{R}^n$, which will be our main concern in this project, then we have only 4 symmetric bilinear forms, and they are:

(i) The trivial one, $B(v, w) = 0 \;\; \forall v, w \in \mathbb{R}^n$.

(ii) $B(v, w) = \sum_{i=1}^{n} x_i y_i$, for $v = (x_1, x_2, ..., x_n), w = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.

(iii) $B(v, w) = -\sum_{i=1}^{n} x_i y_i$, for $v = (x_1, x_2, ..., x_n), w = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.

(iv) $B(v, w) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$, for $1 \leq p < p+q \leq n$, $v = (x_1, x_2, ..., x_n)$, $w = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.

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We are now ready to define the Clifford Algebra associated with V and the quadratic form Q.

Definition. A *Clifford algebra associated with* V *and* Q, denoted $\mathbf{Cliff}(V,Q)$, is a real algebra, together with an injective linear map $i : V \longrightarrow \mathbf{Cliff}(V,Q)$, satisfying the condition $\boxed{(i(v))^2 = Q(v)\cdot 1 \ \ \forall v \in V}^{(*)}$ and so that for any real algebra A, and for every linear map $f : V \longrightarrow A$ with $(f(v))^2 = Q(v)\cdot 1$ $\forall v \in V$ there exists a unique algebra homomorphism $\overline{f} : \mathbf{Cliff}(V,Q) \longrightarrow A$ so that the following diagram is commutative:

$$\begin{tikzcd}
V \arrow[r, "i"] \arrow[dr, "f"'] & \mathbf{Cliff}(V,Q) \arrow[d, "\overline{f}"] \\
 & A
\end{tikzcd}$$

Remarks.

(1.1) The above definition shows that $\mathbf{Cliff}(V,Q)$, if it exists, is unique up to isomorphism of real algebras. To see that this is so, suppose that we have two Clifford algebras $\mathbf{Cliff}_1(V,Q)$, $\mathbf{Cliff}_2(V,Q)$ associated with V and Q, then, by definition, the following diagram is commutative:

$$\begin{tikzcd}
V \arrow[r, "i_1"] \arrow[dr, "i_2"'] & \mathbf{Cliff}_1(V,Q) \arrow[d, shift left, "\overline{f}"] \\
 & \mathbf{Cliff}_2(V,Q) \arrow[u, shift left, "\overline{g}"]
\end{tikzcd}$$

Since both i_1 and i_2 are injective, $\overline{f}$ and $\overline{g}$ are injective and so inverses of each other.

(1.2) To establish existence, we just use another definition of $\mathbf{Cliff}(V,Q)$ and then show that this definition and the above one are equivalent. I will not do this here though!

It can be shown that if one takes $T(V)$ to be the tensor algebra of V, which is constructed as the direct sum $T(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$, then one gets that the Clifford algebra $\mathbf{Cliff}(V,Q) = T(V) \diagup I$, where I is the ideal $I = \{v \otimes v - Q(v) \cdot 1 : v \in V\}$. That is, I is the ideal of $T(V)$ generated by elements of the form $v \otimes v - Q(v) \cdot 1$.

Note that if one sets $J = \{v \otimes v : v \in V\}$, then one gets that $T(V) \diagup J = \Lambda^* V$, the exterior algebra of V.
Actually, most of the interesting algebras that one gets from V are constructed in this fashion.

(1.3) Note that $T(V)$ is $\mathbb{Z} - graded$, and since I is generated by quadratic elements, $\mathbf{Cliff}(V,Q)$ will be $\mathbb{Z}_2 - graded$,
$\boxed{\mathbf{Cliff}(V,Q) = \mathbf{Cliff}^0(V,Q) \oplus \mathbf{Cliff}^1(V,Q)}^{(**)}$. We will come back to this soon and we will define $\mathbf{Cliff}^0(V,Q)$ and $\mathbf{Cliff}^1(V,Q)$, but let us keep them vague for the time being.

(1.4) If one takes the trivial bilinear symmetric form on V, then one gets the trivial quadratic form $Q(v) \equiv 0$ and in this case, $\mathbf{Cliff}(V,Q) \cong \Lambda^* V$.

(1.5) If we pick $u, v \in V$, then we get the following:

$$\boxed{\mathrm{Q(u+v)=B(u+v,u+v)==B(u,u)+2B(u,v)+B(v,v)=Q(u)+2B(u,v)+Q(v)}}^{(1)}.$$

Thus, since i is linear,

$$\boxed{(i(u+v))^2 = (i(u))^2 + (i(v))^2 + i(u) \cdot i(v) + i(v) \cdot i(u)}^{(2)}.$$

Using the fundamental relation $(*)$ above, we get the following:
$(2) \Longrightarrow Q(u+v)\cdot 1 = Q(u)\cdot 1 + Q(v)\cdot 1 + i(u)\cdot i(v) + i(v)\cdot i(u)$
$\Longrightarrow i(u)\cdot i(v) + i(v)\cdot i(u) = Q(u+v)\cdot 1 - Q(u)\cdot 1 - Q(v)\cdot 1$
$= (Q(u+v) - Q(u) - Q(v))\cdot 1 \overset{(1)}{=} 2B(u,v)\cdot 1.$

Thus, $\boxed{i(u)\cdot i(v) + i(v)\cdot i(u) = 2B(u,v)\cdot 1 \quad \forall u,v \in V}^{(3)}$.

(1.6) The relation (3) is sometimes written as $u\cdot v + v\cdot u = 2B(u,v)$ if no confusion would arise.
After all, i is injective. Also, some authors take this relation as the fundamental relation in the definition of the Clifford algebra.

(1.7) Let us get our hands dirty now. Suppose that $\{v_1, v_2, ..., v_n\}$ is an orthonormal basis of V with respect to the bilinear form B, i.e., $B(v_i, v_j) = \delta_{ij}$. Then relation (3) tells us that:

$$v_i\cdot v_j + v_j\cdot v_i = \left\{\begin{array}{cc} 0 & \text{if } i\neq j \\ 2Q(v_i) & \text{if } i=j \end{array}\right\}$$

That is, if $i\neq j$, then $v_i\cdot v_j = -v_j\cdot v_i$.

(1.8) For the special case when $V=\mathbb{R}^n$ and $B(v,w) = -\sum_{i=1}^{n} x_i y_i$, we note the following:
(i) $Q(v) = -\|v\|^2 \quad \forall v\in\mathbb{R}^n$.
(ii) We denote $\mathbf{Cliff}(\mathbb{R}^n, Q)$ in this case by $\mathbf{Cliff}(n)$.
(iii) If we choose an orthonormal basis for $\mathbb{R}^n$ with respect to B, say $\{e_1, e_2, ..., e_n\}$, then we get that $\mathbf{Cliff}(n)$ is generated by the $e_i's$ with the fundamental relation

$$e_i\cdot e_j + e_j\cdot e_i = -2\delta_{ij} \quad \text{for } 1\leq i,j\leq n.$$

That is, $\boxed{\text{if } i \neq j, \text{ then } e_i \cdot e_j = -e_j \cdot e_i}^{(4)}$ and $\boxed{\text{if } i = j, \text{ then } e_i^2 = -1}^{(5)}$.

(1.9) If no confusion would arise, we will drop the $\cdot$ that we have in relations (4) and (5).

(1.10) A basis for **Cliff**(n) may be obtained as follows:
Every element of **Cliff**(n) can be written as a linear combination of finite strings of the form $e_{i_1}e_{i_2}\cdots e_{i_m}$.
Using (4) and (5) repeatedly, this string reduces to a string in which $i_1 < i_2 < \cdots < i_k$.
Here, we used k instead of m to shed light on the fact that some of the $i_l's$ might be the same in which case we eliminate them, using relation (5), after we get them next to each other using relation (4).
For example, something like $e_1e_3e_1e_2$ reduces to $-e_2e_3$, something like $e_1e_3e_2e_4e_3e_3e_1e_2e_3e_4$ reduces to -1, and something like $e_1e_2e_3e_1e_2e_3$ reduces to 1. (Check!).
Hence, it is relatively clear that a basis for **Cliff**(n) is given by: (here, $1 \leq i < j < k \leq n$)

$$\boxed{\begin{array}{c} 1 \\ e_i \\ e_ie_j \\ e_ie_je_k \\ \vdots \\ e_1e_2e_3\cdots e_n \end{array}}$$

(1.11) Note that there are $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$ elements in the above basis.

Hence, $\dim(\mathbf{Cliff}(n)) = 2^n$.

(1.12) Let us compute explicitly some Clifford algebras.

(i) $\mathbf{Cliff}(0)$ is a $2^0 = 1-$dimensional real algebra with basis $\{1\}$. Thus, $\boxed{\mathbf{Cliff}(0) = \mathbb{R}}$.

(ii) $\mathbf{Cliff}(1)$ is a $2^1 = 2-$dimensional real algebra with basis $\{1, e_1\}$, where $e_1^2 = -1$.

Thus, the multiplication table is:

$\cdot$	$\mathbf{1}$	$\mathbf{e_1}$
$\mathbf{1}$	1	e_1
$\mathbf{e_1}$	e_1	-1

If we identify e_1 with the complex number i, since $i^2 = -1$, we get that $\boxed{\mathbf{Cliff}(1) = \mathbb{C}}$.

(iii) $\mathbf{Cliff}(2)$ is a $2^2 = 4-$dimensional real algebra with basis $\{1, e_1, e_2, e_1e_2\}$, where $e_1^2 = -1$, $e_2^2 = -1$, $(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 = -1$, and $e_1e_2 = -e_2e_1$. Thus, the multiplication table is:

$\cdot$	$\mathbf{1}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_1e_2}$
$\mathbf{1}$	1	e_1	e_2	e_1e_2
$\mathbf{e_1}$	e_1	-1	e_1e_2	$-e_2$
$\mathbf{e_2}$	e_2	$-e_1e_2$	-1	e_1
$\mathbf{e_1e_2}$	e_1e_2	e_2	$-e_1$	-1

If one identifies e_1 with i, e_2 with j, and e_1e_2 with k, then one gets that $\boxed{\mathbf{Cliff}(2) = \mathbb{H}}$, where $\mathbb{H}$ is the quaternions. Recall that the quaternions have the multiplication table:

$\cdot$	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

(iv) $\mathbf{Cliff}(3)$ is a $2^3 = 8-$dimensional real algebra with basis $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$, where $e_1^2 = -1$, $e_2^2 = -1$, $e_3^2 = -1$, $(e_1e_2)^2 = -1$, $(e_1e_3)^2 = -1$, $(e_2e_3)^2 = -1$, $(e_1e_2e_3)^2 = 1$, $e_ie_j = -e_je_i$, and so on. Thus, the multiplication table is:

$\cdot$	$\mathbf{1}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_1e_2}$	$\mathbf{e_1e_3}$	$\mathbf{e_2e_3}$	$\mathbf{e_1e_2e_3}$
$\mathbf{1}$	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
$\mathbf{e_1}$	e_1	-1	e_1e_2	e_1e_3	$-e_2$	$-e_3$	$e_1e_2e_3$	$-e_2e_3$
$\mathbf{e_2}$	e_2	$-e_1e_2$	-1	e_2e_3	e_1	$-e_1e_2e_3$	$-e_3$	e_1e_3
$\mathbf{e_3}$	e_3	$-e_1e_3$	$-e_2e_3$	-1	$e_1e_2e_3$	e_1	e_2	$-e_1e_2$
$\mathbf{e_1e_2}$	e_1e_2	e_2	$-e_1$	$e_1e_2e_3$	-1	e_2e_3	$-e_1e_3$	$-e_3$
$\mathbf{e_1e_3}$	e_1e_3	e_3	$-e_1e_2e_3$	$-e_1$	$-e_2e_3$	-1	e_1e_2	$-e_2$
$\mathbf{e_2e_3}$	e_2e_3	$e_1e_2e_3$	e_3	$-e_2$	e_1e_3	$-e_1e_2$	-1	$-e_1$
$\mathbf{e_1e_2e_3}$	$e_1e_2e_3$	$-e_2e_3$	e_1e_3	$-e_1e_2$	$-e_3$	e_2	$-e_1$	1

A good reason for us to be sad at this point is that we wished this would come out to be the octonions, but it is not. Recall that the octonions are not associative. This actually comes out to be $\mathbb{H} \oplus \mathbb{H}$.
You can immediately see a copy of $\mathbb{H}$ in the above table by taking the 1st, 4th, 5th and 6th rows with the 1st, 4th, 5th and 6th columns. This is what we will call the even subalgebra of the Clifford algebra and we will denote it by $\mathbf{Cliff}^0(\mathbf{3})$. (Look at remark 1.3).
Hence, $\boxed{\mathbf{Cliff}(3) = \mathbb{H} \oplus \mathbb{H}}$.

(1.13) Using commutativity of diagrams, we can show that every Clifford algebra $\mathbf{Cliff}(V, Q)$ has a unique canonical anti-automorphism
$\beta : \mathbf{Cliff}(V, Q) \longrightarrow \mathbf{Cliff}(V, Q)$ and a unique canonical automorphism
$\alpha : \mathbf{Cliff}(V, Q) \longrightarrow \mathbf{Cliff}(V, Q)$ such that: (here, we identify $i(v)$ with v)

(i) $\beta(x \cdot y) = \beta(y) \cdot \beta(x)$ $\forall x, y \in \mathbf{Cliff}(V, Q)$, $\beta^2 \equiv id$, and $\beta(v) = v$ $\forall v \in V$.
(ii) $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ $\forall x, y \in \mathbf{Cliff}(V, Q)$, $\alpha^2 \equiv id$, and $\alpha(v) = -v$ $\forall v \in V$.

(1.14) Let us describe the anti-automorphism β, and the automorphism α, in remark (1.13), in terms of their action on an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of V with respect to the bilinear form B.

· β is completely characterized by:
$\beta(1) = 1$, $\beta(e_i) = e_i$, and $\beta(e_{i_1} e_{i_2} \cdots e_{i_k}) = e_{i_k} \cdots e_{i_2} e_{i_1}$.
(This is why it is called an anti-automorphism)
· α is completely characterized by:
$\alpha(1) = 1$, $\alpha(e_i) = -e_i$, and $\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) = (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k}$.

(1.15) Set $\mathbf{Cliff}^0(V, Q) = \{x \in \mathbf{Cliff}(V, Q) : \alpha(x) = x\}$ and $\mathbf{Cliff}^1(V, Q) = \{x \in \mathbf{Cliff}(V, Q) : \alpha(x) = -x\}$.
Observe that $\mathbf{Cliff}(V, Q) = \mathbf{Cliff}^0(V, Q) \oplus \mathbf{Cliff}^1(V, Q)$ and that if $x \in \mathbf{Cliff}^i(V, Q)$, $y \in \mathbf{Cliff}^j(V, Q)$, for $i, j \in \{0, 1\}$, then
$\boxed{x \cdot y \in \mathbf{Cliff}^{i+j(\mathrm{mod}2)}(V, Q)}^{(***)}$.

(Note that $\oplus$ is a direct product of real algebras, not vector spaces).
This is why we say that $\mathbf{Cliff}(V, Q)$ is $\mathbb{Z}_2 - graded$.

(1.16) A basis for $\mathbf{Cliff}^0(V, Q)$ is given by 1 and completely reduced strings of even length (i.e., $e_1 e_2 \cdots e_{2k}$).
A basis for $\mathbf{Cliff}^1(V, Q)$ is given by completely reduced strings of odd length (i.e., $e_1 e_2 \cdots e_{2k-1}$).
Note that, by property $(***)$ in remark 1.15, $\mathbf{Cliff}^0(V, Q)$ is a subalgebra of $\mathbf{Cliff}(V, Q)$, whereas $\mathbf{Cliff}^1(V, Q)$ is not. Also, note that $\mathbf{Cliff}^0(n+1) \cong \mathbf{Cliff}(n)$.

(1.17) We will make up for the torment that we caused in part (iv) of re-

mark (1.12) by stating the Bott's periodicity theorem. At this point, it is not clear what happens if we continue with our construction of **Cliff**(4), **Cliff**(5), **Cliff**(6), ...
We get the following table: (In the table, something like $\mathbb{R}(8)$ means the space of 8×8 real matrices)

n	**Cliff**(n)
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(2)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$
8	$\mathbb{R}(16)$

Bott's Periodicity Theorem. **Cliff**$(n+8) =$ **Cliff**$(n) \otimes$ **Cliff**(8) for $n \geq 1$.

Thus, together with the table above, we know all of the real Clifford algebras over $\mathbb{R}^n$ for each $n \geq 1$, under the bilinear symmetric form $B(v, w) = -\sum_{i=1}^{n} x_i y_i$.

(1.18) Theorem. Let V be a finite dimensional real vector space with a bilinear symmetric form B. Suppose that $V = V_1 \oplus V_2$, where $B(v_1, v_2) = 0$ for all $v_1 \in V_1$ and all $v_2 \in V_2$. Then

$$\mathbf{Cliff}(V, Q) \cong \mathbf{Cliff}(V_1, Q) \otimes \mathbf{Cliff}(V_2, Q).$$

Proof. We will show that **Cliff**$(V_1, Q) \otimes$ **Cliff**(V_2, Q) has the same universal property that characterizes **Cliff**(V, Q). Suppose that we are given a linear map $f : V \longrightarrow A$, where A is any real algebra such that $(f(v))^2 = Q(v) \cdot 1$ $\forall v \in V$. Look at the diagram below.

Let $f_1 = f \circ inc_1 : V_1 \longrightarrow A$ and $f_2 = f \circ inc_2 : V_2 \longrightarrow A$.

$$\begin{tikzcd}
\mathbf{Cliff}(V_1,Q) \arrow[ddrr, "\overline{f_1}"'] & V_1 \arrow[l] \arrow[dr, "inc_1"'] & & V_2 \arrow[dl, "inc_2"] \arrow[r] & \mathbf{Cliff}(V_2,Q) \arrow[ddll, "\overline{f_2}"] \\
& & V \arrow[d, "f"] & & \\
& & A & &
\end{tikzcd}$$

It is easy to check that $(f_1(v))^2 = Q(v)\cdot 1 \ \ \forall v \in V_1$ and that $(f_2(v))^2 = Q(v)\cdot 1$ $\forall v \in V_2$. Thus, by the universal property of $\mathbf{Cliff}(V_1, Q)$ and $\mathbf{Cliff}(V_2, Q)$, we get unique real algebra homomorphisms $\overline{f_1} : \mathbf{Cliff}(V_1, Q) \longrightarrow A$ and $\overline{f_2}$: $\mathbf{Cliff}(V_2, Q) \longrightarrow A$ such that the diagrams that you expect are commutative. Now, we wish to determine the relation between $\overline{f_1}$ and $\overline{f_2}$. We do this in 3 steps:

Step 1. Let $v_1 \in V_1$ and $v_2 \in V_2$.
Then, in A, we have $\overline{f_1}(v_1)\cdot\ \overline{f_2}(v_2) = -\overline{f_2}(v_2)\cdot\overline{f_1}(v_2)$.
This is because, in this case,
$B(v_1 + v_2, v_1 + v_2) = B(v_1, v_1) + \underbrace{2B(v_1, v_2)}_{=0} + B(v_2, v_2)$
$= B(v_1, v_1) + B(v_2, v_2) = Q(v_1) + Q(v_2)$
$\Longrightarrow f(v_1 + v_2)\cdot f(v_1 + v_2) = Q(v_1 + v_2)\cdot 1 = Q(v_1)\cdot 1 + Q(v_2)\cdot 1 = f(v_1)\cdot f(v_1) + f(v_2)\cdot f(v_2)$.
But $f(v_1{+}v_2)\cdot f(v_1{+}v_2) = f(v_1)\cdot f(v_1) + f(v_1)\cdot f(v_2) + f(v_2)\cdot f(v_1) + f(v_2)\cdot f(v_2)$.
Thus, $f(v_1)\cdot f(v_2) + f(v_2)\cdot f(v_1) = 0$ and so $f(v_1)\cdot f(v_2) = -f(v_2)\cdot f(v_1)$.
Thus, by commutativity of diagrams, we get our result.

Step 2. If $v_1 \in V_1$ and $w_1 \cdot \cdots \cdot w_n \in V_2$.
Then, in A, we have $\overline{f_1}(v_1)\cdot\ \overline{f_2}(w_1 \cdot \cdots \cdot w_n) = (-1)^n\, \overline{f_2}(w_1 \cdot \cdots \cdot w_n)\cdot\overline{f_1}(v_2)$.
This is easy to see by combining what we found in step 1 with the fact that $\overline{f_2}$ is a homomorphism.

Step 3. If $v_1 \cdot \cdots \cdot v_m \in V_1$ and $w_1 \cdot \cdots \cdot w_n \in V_2$.
Then, in A, we have $\overline{f_1}(v_1 \cdot \cdots \cdot v_m)\cdot\ \overline{f_2}(w_1 \cdot \cdots \cdot w_n) = (-1)^{mn}\,\overline{f_2}(w_1 \cdot \cdots \cdot w_n)\cdot \overline{f_1}(v_1 \cdot \cdots \cdot v_m)$.
This is easy to see by combining steps 1 and 2.

Now, define $\overline{f} : \mathbf{Cliff}(V_1, Q) \otimes \mathbf{Cliff}(V_2, Q) \longrightarrow A$ by

$$\overline{f}(x \otimes y) = \overline{f}_1(x) \cdot \overline{f}_2(x) \quad \text{for } x \in \mathbf{Cliff}(V_1, Q),\ y \in \mathbf{Cliff}(V_2, Q).$$

This is clearly a homomorphism of real algebras; it is linear and it is a ring homomorphism.

Now, we replace A by $\mathbf{Cliff}(V, Q)$ and we use the universal property of $\mathbf{Cliff}(V, Q)$ to get that $\overline{f} : \mathbf{Cliff}(V_1, Q) \otimes \mathbf{Cliff}(V_2, Q) \longrightarrow \mathbf{Cliff}(V, Q)$ is an isomorphism. ■

Chapter 2 Spinor Groups

We recall from remarks (1.13) and (1.14) above what the structure maps α and β are. Also, we will assume the reader is familiar with all of what was done in chapter 1.

Remarks.
(2.1) Note that $\alpha|_{\mathbf{Cliff^0(V,Q)}} = I$, and that $\alpha|_{\mathbf{Cliff^1(V,Q)}} = -I$. Also, note that $\alpha\beta = \beta\alpha$ and this can be established by showing that they agree on every element of a basis for $\mathbf{Cliff}(V, Q)$.
Moreover, $\forall x \in \mathbf{Cliff}(V, Q)$, we have $\alpha(x)\beta(x) = \beta(x)\alpha(x) = 1$.

(2.2) Define $\gamma = \alpha\beta = \beta\alpha : \mathbf{Cliff}(V, Q) \longrightarrow \mathbf{Cliff}(V, Q)$.
Then γ is completely characterized by: $\gamma(1) = 1,\ \gamma(e_i) = -e_i$, and
$\gamma(e_{i_1} e_{i_2} \cdots e_{i_k}) = (-1)^k e_{i_k} e_{i_{k-1}} \cdots e_{i_1}$.

(2.3) Let us compute the maps α, β, and γ for $\mathbf{Cliff}(1) = \mathbb{C}$, $\mathbf{Cliff}(2) = \mathbb{H}$, and $\mathbf{Cliff}(\mathbf{3}) = \mathbb{H} \oplus \mathbb{H}$.
One gets the following three tables, respectively:

	$\mathbf{1}$	$\mathbf{i}$
α	1	$-i$
β	1	i
γ	1	$-i$

,

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
α	1	$-i$	$-j$	k
β	1	i	j	$-k$
γ	1	$-i$	$-j$	$-k$

, and

	$\mathbf{1}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_1e_2}$	$\mathbf{e_1e_3}$	$\mathbf{e_2e_3}$	$\mathbf{e_1e_2e_3}$
α	1	$-e_1$	$-e_2$	$-e_3$	e_1e_2	e_1e_3	e_2e_3	$-e_1e_2e_3$
β	1	e_1	e_2	e_3	$-e_1e_2$	$-e_1e_3$	$-e_2e_3$	$-e_1e_2e_3$
γ	1	$-e_1$	$-e_2$	$-e_3$	$-e_1e_2$	$-e_1e_3$	$-e_2e_3$	$e_1e_2e_3$

.

Note that in the 3 cases, $x\gamma(x) = \gamma(x)x = 1 \ \forall x$. This resembles the action of *conjugation*.

(2.4) From now on, if no confusion would arise, we shall write $\overline{x}$ for $\gamma(x)$.

(2.5) Note that elements of V can be thought of as the span of $1-$vectors in $\mathbf{Cliff}(V, Q)$.
This is by injectivity of i.

(2.6) Definition. We define the *Pinor group*, denoted $\mathbf{Pin}(V, Q)$, by:

$$\boxed{\mathbf{Pin}(V, Q) = \{x \in \mathbf{Cliff}(V, Q) : x\overline{x} = 1, \text{ and } xV\,(\beta x) \subset V\}}.$$

We also define the Spinor group, denoted $\mathbf{Spin}(V, Q)$, by:

$$\boxed{\begin{array}{l}\mathbf{Spin}(V, Q) = \mathbf{Pin}(V, Q) \cap \mathbf{Cliff}^0(V, Q) = \\ \{x \in \mathbf{Cliff}^0(V, Q) : x\overline{x} = 1, \text{ and } xV\,(\beta x) \subset V\}\end{array}}.$$

(2.7) The condition $xV\,(\beta x) \subset V$ means that there is an action of the Clifford algebra, $\mathbf{Cliff}(V,Q)$, on V given by multiplication on the left by x and on the right by βx. The action may be written as: $\pi x : \mathbf{Cliff}(V,Q) \times V \longrightarrow V$ given by $\pi x(v) = x \cdot v = xv\,(\beta x)$. It is an action by the associativity of the Clifford Algebra and the fact that elements of V can be realized as the 1-vectors in $\mathbf{Cliff}(V,Q)$.
$(1{\cdot}v = 1v1 = v,\ (xy){\cdot}v = xyv\,(\beta xy) = xyv\beta y\beta x = x\,(yv\beta y)\,\beta x = x\,(y \cdot v)\,\beta x = x \cdot (y \cdot v))$.

(2.8) Let us compute $\mathbf{Pin}(1)$, $\mathbf{Spin}(1)$, $\mathbf{Pin}(\mathbf{2})$, and $\mathbf{Spin}(2)$.

$\mathbf{Pin}(1) = \{x \in \mathbf{Cliff}(1) : x\overline{x} = 1, \text{ and } x\mathbb{R}\,(\beta x) \subset \mathbb{R}\}$
$= \{x \in \mathbb{C} : x\overline{x} = 1, \text{ and } x\mathbb{R}\,(\beta x) \subset \mathbb{R}\}$
$= \{x = a + ib \in \mathbb{C} : \|x\| = 1, \text{ and } x^2 \in \mathbb{R}\}$
$= \{x = a + ib \in \mathbb{C} : a^2 + b^2 = 1, \text{ and either } a \text{ or } b = 0\}$
$= \{\pm 1, \pm i\}$.

$\mathbf{Spin}(1) = \mathbf{Pin}(1) \cap \mathbf{Cliff}^{\mathbf{0}}(\mathbf{1})$
$= \{\pm 1, \pm i\} \cap \mathbb{R}$
$= \{\pm 1\}$.

$\mathbf{Pin}(2) = \{x \in \mathbf{Cliff}(2) : x\overline{x} = 1, \text{ and } x(\text{ 1-vectors})\,(\beta x) \subset (\text{1-vectors})\}$.
Recall that $\mathbf{Cliff}(2) = \mathbb{H}$ has basis $\{1, i, j, ij = k\}$.
If $x \in \mathbf{Cliff}(2)$, then $x = a1 + bi + cj + dk$ and the set of 1-vectors in this case is spanned by i and j.
Thus, if $v = a'i + b'j \in V = \mathbb{R}^2$ is a 1-vector, then
$xv\,(\beta x) = (a1 + bi + cj + dk)\,(a'i + b'j)\,(a1 + bi + cj - dk) \in V \overset{\text{check}}{\Longleftrightarrow} b = c = 0$ or $a = d = 0$.
Thus, $\mathbf{Pin}(2) = \{x = a1 + bi + cj + dk \in \mathbf{Cliff}(2) : x\overline{x} = 1, \text{ and } (b = c = 0 \text{ or } a = d = 0)\} = \{x = a1 + dk : a^2 + d^2 = 1\} \cup \{x = bi + cj : b^2 + c^2 = 1\}$.

$\mathbf{Spin}(2) = \mathbf{Pin}(2) \cap \mathbf{Cliff}^0(2)$
$= \big(\{x = a1 + dk : a^2 + d^2 = 1\} \cup \{x = bi + cj : b^2 + c^2 = 1\}\big) \cap \mathbb{C}$
$= \big(\{x = a1 + dk : a^2 + d^2 = 1\} \cap \mathbb{C}\big) \cup \varnothing$
$= S^1 = U(1)$.

(2.9) Let us see, geometrically, what the action of $\mathbf{Spin}(2)$ on $\mathbb{R}^2$ is.
For this, one takes an element $x = a1 + dk \in \mathbf{Spin}(2)$, where, of course, $a^2 + d^2 = 1$.
Then one writes $x = a1 + dk = \cos\theta + \sin\theta k$, $\theta \in \mathbb{R}$. (Here, one drops 1 for simplicity).
One also takes an ideal element $v = bi + cj \in \mathbb{R}^2$. (Here, again, one identifies $\mathbb{R}^2$ with 1-vectors).
Now, from remark 2.7, one computes $\pi x(v)$.
This gives $\pi x(v) = x \cdot v = xv\,(\beta x) = (\cos\theta + \sin\theta k)\,(bi + cj)\,(\cos\theta - \sin\theta k)$
$= (b\cos\theta i + c\cos\theta j + b\sin\theta j - c\sin\theta i)\,(\cos\theta - \sin\theta k)$
$= ((b\cos\theta - c\sin\theta)\,i + (c\cos\theta + b\sin\theta)j)\,(\cos\theta - \sin\theta k)$
$= (b\cos 2\theta - c\sin 2\theta)i + (b\sin 2\theta + c\cos 2\theta)j.$

Thus, π is a rotation by 2θ. In matrix language, this reminds us of
$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$.

(2.10) In the following remarks, (2.11) to (2.17), we will prove an extremly marvellous result that will make our work, up to this point, worthwhile.
The result states that the Lie algebra of $\mathbf{Spin}(n)$, $\mathfrak{spin}(n)$, is isomorphic to the Lie algebra $\mathfrak{so}(n)$.
That is, $\boxed{\mathfrak{spin}(n) \cong \mathfrak{so}(n)}$.
Throughout the proof, we will assume the fact that both $\mathbf{Pin}(V, Q)$ and $\mathbf{Spin}(V, Q)$ are closed subgroups of the invertible elements of $\mathbf{Cliff}(V, Q)$ and hence are Lie groups themselves.

(2.11) The map $\pi : \mathbf{Pin}(V, Q) \longrightarrow O(V, Q)$ given by $(\pi x)v = xv\beta(x)$ is a well-defined group homomorphism.

Proof. Here, by well-defined, we mean that the bilinear symmetric form B is preserved under πx and thus, $\pi x \in O(V, Q)$. To see this, we compute $B((\pi x)\,v, (\pi x)\,v)$.

Note that $Q(v) = B(v,v) = v^2 = -v\alpha(v) \quad \forall v \in V$. Also, since $x \in \mathbf{Pin}(V,Q)$, $(\pi x)\, v \in V \quad \forall v \in V$.

Thus,
$B((\pi x)\, v, (\pi x)\, v) = Q\,((\pi x)\, v) = -\,(\pi x)\, v\alpha((\pi x)\, v) = -xv\beta(x)\alpha(xv\beta(x)) =$
$-\,xv\underbrace{\beta(x)\alpha(x)}_{=1}\underbrace{\alpha(v)}_{=-v}\gamma(x) = xv^2\gamma(x) = x\underbrace{B(v,v)}_{\in\mathbb{R}}\gamma(x) = B(v,v)x\gamma(x) = B(v,v)$.
Thus, $\pi x \in O(V,Q)$.

We already have shown that π is a group homomorphism in remark 2.7. ∎

(2.12) $\ker \pi = \{\pm 1\}$.

Proof. $\ker \pi = \{x \in \mathbf{Pin}(V,Q) : (\pi x)\, v = v \quad \forall v \in V\} = \{x \in \mathbf{Pin}(V,Q) : xv\beta(x) = v \quad \forall v \in V\} = \{x \in \mathbf{Pin}(V,Q) : xv\beta(x)\alpha(x) = v\alpha(x) \quad \forall v \in V\} = \{x \in \mathbf{Pin}(V,Q) : xv = v\alpha(x) \quad \forall v \in V\}$.
It is not hard to show that if $xv = v\alpha(x) \quad \forall v \in V$, then $x \in \mathbb{R}$.
Thus, $= \{x \in \mathbf{Pin}(V,Q) \cap \mathbb{R}\} = \{\pm 1\}$. ∎

(2.13) Let $i < j$. Then $(e_ie_j)(e_ie_j) = -1$ and so we can define $x(t) = e^{(e_ie_j)t} = \cos t + (\sin t)e_ie_j$.
We claim that $x(t) \in \mathbf{Spin}(V,Q)$ and $\pi(x(t)) =$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array} \\
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & & \cos 2t & & & & -\sin 2t & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
 & & & \sin 2t & & & & \cos 2t & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{pmatrix}
\end{array}.$$

Proof. $x(t) = \cos t + (\sin t)e_ie_j$ is a 2-vector and so it lies in $\mathbf{Cliff}^0(V,Q)$.
$x(t)\overline{x(t)} = (\cos t + (\sin t)e_ie_j)(\cos t - (\sin t)e_ie_j) = \cos^2 t + \sin^2 t = 1$.
Also, if we take any 1-vector $v = e_k$ in the basis for V, then we get that

$$x(t)v\beta(x(t)) = (\cos t + (\sin t)e_ie_j)e_k(\cos t - (\sin t)e_ie_j) = \left\{ \begin{array}{cl} e_k & \text{if } k \notin \{i,j\} \\ (\cos 2t + (\sin 2t)e_ie_j)e_k & \text{if } k \in \{i,j\} \end{array} \right\}.$$

Note that $(\cos 2t + (\sin 2t)e_ie_j)e_k$ is also a 1-vector.
Thus, $x(t)V\beta(x(t)) \subset V$ and so $x(t) \in \mathbf{Pin}(V,Q)$.

Therefore, $x(t) \in \mathbf{Cliff}^0(V,Q) \cap \mathbf{Pin}(V,Q) = \mathbf{Spin}(V,Q)$.

If we write that matrix for $\pi(x(t))$ with respect to the basis $\{e_1, e_2, ..., e_n\}$ of V, which we did 4 lines above, then we get the matrix given in the statement of the result. ■

(2.14) The elements $\frac{d}{dt}|_{t=0}(\pi(x(t))) = \begin{pmatrix} 0 & & & & \\ & \ddots & & -2 & \\ & & 0 & & \\ & 2 & & \ddots & \\ & & & & 0 \end{pmatrix}$ form a basis for $\mathfrak{so}(n)$.

Proof. $\mathfrak{so}(n) = \{n \times n \text{ real matrices } X : X^{tr} = -X\} = \{n \times n \text{ real matrices } X = (a_{ij}) : a_{ij} = -a_{ji}\} = \{n \times n \text{ real matrices } X = (a_{ij}) : a_{ii} = 0 \text{ and } a_{ij} = -a_{ji}\}$. ■

(2.15) The elements $\frac{d}{dt}|_{t=0}(x(t)) = \frac{d}{dt}|_{t=0}(\cos t + (\sin t)e_ie_j) = e_ie_j,\ i < j$, which belong to the Lie algebra $\mathfrak{spin}(n)$, map onto the elements in remark 2.14. Thus, π is surjective.

Proof. This was "somehow" done in remark 2.13. ■

(2.16) The Lie algebra $\mathfrak{g}$ spanned by elements of the form $e_i e_j,\ i < j$, with $[e_i e_j, e_k e_l] = e_i e_j e_k e_l - e_k e_l e_i e_j$, is isomorphic to the Lie algebra $\mathfrak{so}(n)$.

Proof. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ elements in the basis for $\mathfrak{g}$. Also, $\dim(\mathfrak{so}(n)) = \frac{n(n-1)}{2}$.
Let us look at $\mathfrak{g}$ more closely. The bracket has some interesting properties.

- If i, j, k and l are different, then
  $[e_i e_j, e_k e_l] = e_i e_j e_k e_l - e_k e_l e_i e_j = e_i e_j e_k e_l - e_i e_j e_k e_l = 0$.
- If i, k and l are different, then
  $[e_i e_l, e_k e_l] = e_i e_l e_k e_l - e_k e_l e_i e_l = e_i e_k - e_k e_i = 2 e_i e_k$.
- If $i = k$ and $j = l$, then
  $[e_i e_j, e_k e_l] = [e_i e_j, e_i e_j] = e_i e_j e_i e_j - e_i e_j e_i e_j = 0$.

It is easy to see that the bracket satisfies Jacobi identity. Thus, $\mathfrak{g}$ is indeed a Lie algebra.

Let us show that $\pi : \mathfrak{g} \longrightarrow \mathfrak{so}(n)$ preserves the bracket, and hence, since the dimensions are the same, a Lie algebra isomorphism. We will only do it for the nontrivial bracket case above.
The other two cases are immediate. So suppose that we have the two elements $e_i e_j$ and $e_k e_j$.
They will be mapped under π to the two elements X_{ij} and $X_{kj} \in \mathfrak{so}(n)$, where X_{ij} has all entries 0 except the ij and ji entries in which we have -2 and 2, respectively, and X_{kj} has all entries 0 except the kj and jk entries in which we have -2 and 2, respectively.
One checks that $[X_{ij}, X_{kj}] = X_{ij} X_{kj} - X_{kj} X_{ij} = 2 X_{ik}$, where X_{ik}, of course, has all entries 0 except the ik and ki entries in which we have -2 and 2, respectively.
Note that $e_i e_k \stackrel{\pi}{\longmapsto} X_{ik}$. ■

(2.17) $\pi : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ is an isomorphism since $\ker \pi$ is finite. Thus, a basis for the Lie algebra $\mathfrak{spin}(n)$ is given by the span of 2-vectors $e_i e_j$, with $i < j$. Hence, $\boxed{\mathfrak{spin}(n) \cong \mathfrak{so}(n)}$.

Chapter 3 Construction of the Lie algebra $\mathfrak{e}_8$

We begin by introducing an inner product on $\mathfrak{spin}(n)$, then compute the roots and find the Dynkin diagram.
For E_8, there is no representation of smaller degree than the adjoint representation Ad, so we use that.

(3.1) We identified elements of $\mathfrak{spin}(n)$ with skew-symmetric matrices, in $\mathfrak{so}(n)$, of the form given in (2.14). The 2-vector $e_i e_j$ was identified with the matrix X_{ij}, where X_{ij} has all entries 0 except the ij and ji entries in which we have -2 and 2, respectively.

On $\mathfrak{so}(n)$, we use the inner product $\langle X, Y \rangle = trace(XY)$.

Thus, we define $\langle , \rangle_{\mathfrak{spin}(n)} : \mathfrak{spin}(n) \times \mathfrak{spin}(n) \longrightarrow \mathbb{R}$ by the formula

$$\langle e_i e_j, e_k e_l \rangle_{\mathfrak{spin}(n)} = \langle X_{ij}, X_{kl} \rangle = trace(X_{ij} X_{kl}).$$

This gives $\langle e_i e_j, e_i e_j \rangle_{\mathfrak{spin}(n)} = \langle X_{ij}, X_{ij} \rangle = trace(X_{ij} X_{ij}) = -8$.

To remove the undesirable factor -8, let us instead define $\langle X_{ij}, X_{ij} \rangle = \frac{-1}{8} trace(X_{ij} X_{ij})$. Hence,

$$\boxed{\langle e_i e_j, e_k e_l \rangle_{\mathfrak{spin}(n)} = \langle X_{ij}, X_{kl} \rangle = \frac{-1}{8} trace(X_{ij} X_{kl}) = \delta_{ik}\delta_{jl}}.$$

(3.2) Following [2], we let x_i denote the 2×2 block diagonal matrix whose *i-th* diagonal block is $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ and whose other diagonal blocks are zero. Then the roots of $\mathfrak{so}(2n)$ are the matrices $\pm x_i \pm x_j$, $1 \leq i < j \leq n$. Note that $\langle -x_i + x_j, -x_i + x_j \rangle = 2$.

Also, the inner product of $\pm x_i \pm x_j$ with $\pm x_k \pm x_l$ is 0 if $\{i,j\} \cap \{k,l\} = \varnothing$, and is ± 1 if $\{i,j\} \cap \{k,l\}$ is one element. Moreover, the roots $-x_i + x_j$ and $x_i + x_j$ are orthogonal.

If we consider the special case $\mathfrak{spin}(16)$, then, as a base, we may take the 8 roots: (see [2])

1	$-x_1 + x_2$
2	$-x_2 + x_3$
3	$-x_3 + x_4$
4	$-x_4 + x_5$
5	$-x_5 + x_6$
6	$-x_6 + x_7$
7	$-x_7 + x_8$
8	$x_7 + x_8$

Thus, the Dynkin diagram for $\mathfrak{spin}(16)$ is:

$$\begin{array}{ccccccccccccc} & & & & & & & & & & \overset{7}{\mathbf{O}} & & \\ & & & & & & & & & & | & & \\ \underset{1}{\mathbf{O}} & - & \underset{2}{\mathbf{O}} & - & \underset{3}{\mathbf{O}} & - & \underset{4}{\mathbf{O}} & - & \underset{5}{\mathbf{O}} & - & \underset{6}{\mathbf{O}} & - & \underset{8}{\mathbf{O}} \end{array}$$

(3.3) We shall use the following theorem without proof. The proof can be found in [1].

Theorem. The algebra $\mathbf{Cliff}^0(V,Q)$ is semisimple and so all of its representations are completely reducible. Moreover, if $\dim(V) = 4n$, then $\mathbf{Cliff}^0(V,Q)$ has two self-dual irreducible representations $\triangle^+$ and $\triangle^-$, of degree 2^{2n-1} affording representations $\triangle^+$ and $\triangle^-$ of $\mathfrak{spin}(4n)$ having weights:

$$\tfrac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_{2n}) \qquad \text{(even number of } - \text{ signs) for } \triangle^+$$

$$\tfrac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_{2n}) \qquad \text{(odd number of } - \text{ signs) for } \triangle^-.$$

Here, the $x_i's$ are as in 3.2.

(3.4) In what follows, we will construct the Lie algebra $\mathfrak{e}_8$. The idea is to first take $\mathfrak{spin}(16)+\triangle^+$ and work with it for a while. This involves defining an action of $\mathfrak{spin}(16)$ on $\triangle^+$, then we choose a nonzero symmetric bilinear map $\langle,\rangle_{\triangle^+}:\triangle^+\times\triangle^+\longrightarrow\mathbb{R}$ that is invariant under $\mathfrak{spin}(16)$. Then we use $\langle,\rangle_{\triangle^+}$ to define a map $[,]:\triangle^+\times\triangle^+\longrightarrow\mathfrak{spin}(16)$ that is invariant under $\mathfrak{spin}(16)$. Then we proceed to give $\mathfrak{spin}(16)+\triangle^+$ an inner product with $\mathfrak{spin}(16)$ and $\triangle^+$ orthogonal. Then we define a Lie bracket on $\mathfrak{spin}(16)+\triangle^+$ and show that $\mathfrak{spin}(16)+\triangle^+$ becomes a Lie algebra with an invariant inner product. After we do this, we proceed to compute the simple roots of $\mathfrak{spin}(16)+\triangle^+$ and show that they have the Dynkin diagram associated to E_8.

(3.5) Since $\triangle^+$ is a representation of $\mathfrak{spin}(16)$ over $\mathbb{R}$, we have that $\forall a\in\mathfrak{spin}(16)$, and $u\in\triangle^+$, $[a,u]\in\triangle^+$ and satisfies the Jacobi identity $[[a,b],u]=[a,[b,u]]-[b,[a,u]]$. This is the action of $\mathfrak{spin}(16)$ on $\triangle^+$ that we wanted to define.

(3.6) Choose a nonzero symmetric bilinear map $\langle,\rangle_{\triangle^+}:\triangle^+\times\triangle^+\longrightarrow\mathbb{R}$ that is invariant under $\mathfrak{spin}(16)$.
By invariant under $\mathfrak{spin}(16)$, we mean that $\forall a\in\mathfrak{spin}(16)$, $\langle[a,u],v\rangle_{\triangle^+}+\langle u,[a,v]\rangle_{\triangle^+}=0$.

(3.7) In the following Lemma, we transpose the action $\mathfrak{spin}(16)\times\triangle^+\longrightarrow\triangle^+$ to get a map $[,]:\triangle^+\times\triangle^+\longrightarrow\mathfrak{spin}(16)$.
Lemma. For all $u,v\in\triangle^+$, there is a unique $[u,v]\in\mathfrak{spin}(16)$ such that

$$\langle a,[u,v]\rangle_{\mathfrak{spin}(16)}=\langle[a,u],v\rangle_{\triangle^+}\quad\forall a\in\mathfrak{spin}(16)\text{ and }[u,v]\text{ is bilinear in }u,v.$$

Moreover, if $v,w\in\mathbb{C}\otimes\triangle^+$ are such that

$$e_{2q-1}e_{2q}v=iv\qquad\text{for all } q=1,2,...,8\quad(\text{corresponding to a weight } \tfrac{1}{2}\sum_{i=1}^{8}x_i),$$

$$e_{2q-1}e_{2q}w=-iw\qquad\text{for all } q=1,2,...,8\quad(\text{corresponding to a weight } -\tfrac{1}{2}\sum_{i=1}^{8}x_i),$$

and $\langle v,w\rangle_{\triangle^+}=1$, then

(i) $\quad[v,w]=i(e_1e_2+e_3e_4+...+e_{15}e_{16});$

(ii) $[e_{2q}e_{2r}v, w] = (e_{2q-1} + ie_{2q})(e_{2r-1} + ie_{2r}),\ q < r;$

(iii) $[e_{2q_1}e_{2q_2}\cdots e_{2q_{2m}}v, w] = 0$ if $m > 1$ and $q_1 < q_2 < \cdots < q_{2m}$.

Proof. $\langle [a, u], v\rangle_{\triangle^+} : \triangle^+ \times \triangle^+ \longrightarrow \mathbb{R}$ can be thought of as a linear functional on $\mathfrak{spin}(16)$ as follows:
Fix $u, v \in \triangle^+$ and define $\varphi : \mathfrak{spin}(16) \longrightarrow \mathbb{R}$ by

$$\varphi(a) = \langle [a, u], v\rangle_{\triangle^+}$$

Then φ is linear in a because for $c_1, c_2 \in \mathbb{R}$ and $a_1, a_2 \in \mathfrak{spin}(16)$, we have:
$\varphi(c_1a_1 + c_2a_2) = \langle [c_1a_1 + c_2a_2, u], v\rangle_{\triangle^+} = \langle [c_1a_1, u] + [c_2a_2, u], v\rangle_{\triangle^+} =$
$= \langle c_1[a_1, u] + c_2[a_2, u], v\rangle_{\triangle^+} = \langle c_1[a_1, u], v\rangle_{\triangle^+} + \langle c_2[a_2, u], v\rangle_{\triangle^+} =$
$= c_1 \langle [a_1, u], v\rangle_{\triangle^+} + c_2 \langle [a_2, u], v\rangle_{\triangle^+} = c_1\varphi(a_1) + c_2\varphi(a_2)$.

Thus, as in [2] page 299, there exists a unique $b \in \mathfrak{spin}(16)$:

$$\varphi(a) = \langle a, b\rangle_{\mathfrak{spin}(16)} \qquad \forall a \in \mathfrak{spin}(16).$$

Define $[,] : \triangle^+ \times \triangle^+ \longrightarrow \mathfrak{spin}(16)$ by: $[u, v] \doteqdot b$, where b is as above.

Since φ is bilinear in u and v, so is b. ■

(i) $[v, w]$ is the unique $b \in \mathbb{C} \otimes \mathfrak{spin}(16)$ such that $\varphi(a) = \langle [a, v], w\rangle_{\triangle^+} = \langle a, b\rangle_{\mathfrak{spin}(16)}$ $\forall a \in \mathfrak{spin}(16)$.

· If $a = e_{2q-1}e_{2q} \in \mathfrak{spin}(16)$, then
$\varphi(a) = \langle [e_{2q-1}e_{2q}, v], w\rangle_{\triangle^+} = \langle e_{2q-1}e_{2q}v, w\rangle_{\triangle^+} = \langle iv, w\rangle_{\triangle^+} = i\,\langle v, w\rangle_{\triangle^+} = i$.
· If $a = e_re_s \in \mathfrak{spin}(16)$, where e_re_s is not of the form $e_{2q-1}e_{2q}$, then $\varphi(a) = 0$.

Thus, the unique b must be $= i(e_1e_2 + e_3e_4 + ... + e_{15}e_{16})$. ■

(ii) Let us do this for e_2e_4. That is, let us show that $[e_2e_4v, w] = (e_1 + ie_2)(e_3 + ie_4)$. $[e_2e_4v, w]$ is the unique $b \in \mathbb{C} \otimes \mathfrak{spin}(16)$ such that $\varphi(a) = \langle [a, e_2e_4v], w\rangle_{\triangle^+} = \langle a, b\rangle_{\mathfrak{spin}(16)}$ $\forall a \in \mathfrak{spin}(16)$.
If $a = e_re_s$, then $\varphi(a) = \langle e_re_se_2e_4v, w\rangle_{\triangle^+} = 0$ except when $(r, s) = (1, 3),\ (1, 4),$ $(2, 3)$, or $(2, 4)$.

· $(r,s)=(1,3) \Longrightarrow \varphi(a)=\langle e_1e_3e_2e_4v,w\rangle_{\triangle^+}=\langle -e_1e_2e_3e_4v,w\rangle_{\triangle^+}$
$=\left\langle -i^2v,w\right\rangle_{\triangle^+}=\langle v,w\rangle_{\triangle^+}=1.$

· $(r,s)=(1,4) \Longrightarrow \varphi(a)=\langle e_1e_4e_2e_4v,w\rangle_{\triangle^+}=\langle e_1e_2v,w\rangle_{\triangle^+}=\langle iv,w\rangle_{\triangle^+}$
$=i\,\langle v,w\rangle_{\triangle^+}=i.$

· $(r,s)=(2,3) \Longrightarrow \varphi(a)=\langle e_2e_3e_2e_4v,w\rangle_{\triangle^+}=\langle e_3e_4v,w\rangle_{\triangle^+}=\langle iv,w\rangle_{\triangle^+}$
$=i\,\langle v,w\rangle_{\triangle^+}=i.$

· $(r,s)=(2,4) \Longrightarrow \varphi(a)=\langle e_2e_4e_2e_4v,w\rangle_{\triangle^+}=\langle -v,w\rangle_{\triangle^+}=-\,\langle v,w\rangle_{\triangle^+}$
$=-1.$

Therefore, the unique b is $=e_1e_3+ie_1e_4+ie_2e_3-e_2e_4=(e_1+ie_2)(e_3+ie_4)$.
■

(iii) It suffices to note that $\forall e_re_s,\ r<s,\ \langle e_re_se_{2q_1}e_{2q_2}\cdots e_{2q_{2m}}v,w\rangle_{\triangle^+}=0.$
■

(3.8) By using invariance of $\langle,\rangle_{\mathfrak{spin}(16)}$ under $\mathfrak{spin}(16)$, invariance of $\langle,\rangle_{\triangle^+}$ under $\mathfrak{spin}(16)$, the definition of $[u,v]$ for $u,v\in\triangle^+$, and properties of the action of $\mathfrak{spin}(16)$ on $\triangle^+$, we can show that :

$$[,]:\triangle^+\times\triangle^+\longrightarrow\mathfrak{spin}(16)\quad\text{is invariant under }\mathfrak{spin}(16).$$

(Here, we need to show that $\forall a\in\mathfrak{spin}(16)$, $[a,[u,v]]=[[a,u]\,,v]+[u,[a,v]]$).

(3.9) Define the inner product $\langle,\rangle_{\mathfrak{e}_8}:(\mathfrak{spin}(16)+\triangle^+)\times(\mathfrak{spin}(16)+\triangle^+)\longrightarrow\mathbb{R}$ by

$$\langle a+u,b+v\rangle_{\mathfrak{e}_8}=\langle a,b\rangle_{\mathfrak{spin}(16)}+\langle u,v\rangle_{\triangle^+}\qquad\forall a,b\in\mathfrak{spin}(16),\ u,v\in\triangle^+.$$

This inner product makes $\mathfrak{spin}(16)$ and $\triangle^+$ orthogonal, it is invariant under $\mathfrak{spin}(16)$ by definition, and it is invariant under $\triangle^+$ by definition of $[u,v]$.

(3.10) Define the Lie bracket $[,]_{\mathfrak{e}_8} : (\mathfrak{spin}(16) + \triangle^+) \times (\mathfrak{spin}(16) + \triangle^+) \longrightarrow \mathfrak{spin}(16) + \triangle^+$ by:

$$[a,b]_{\mathfrak{e}_8} = [a,b]_{\mathfrak{spin}(16)} \quad \forall a,b \in \mathfrak{spin}(16),$$

$$[a,u]_{\mathfrak{e}_8} \text{ as in the action of } \mathfrak{spin}(16) \text{ on } \triangle^+ \;\; \forall a \in \mathfrak{spin}(16),\; u \in \triangle^+,$$

$$[u,a]_{\mathfrak{e}_8} \doteqdot -[a,u]_{\mathfrak{e}_8} \quad \forall a \in \mathfrak{spin}(16),\; u \in \triangle^+,$$

$$[u,v]_{\mathfrak{e}_8} \text{ as in Lemma 3.7} \quad \forall u,v \in \triangle^+.$$

(3.11) We have the following theorem.

Theorem. Under the above Lie bracket, $[,]_{\mathfrak{e}_8}$, $\mathfrak{spin}(16) + \triangle^+$ becomes a Lie algebra.

Proof. We have to establish the following properties of the Lie bracket:

(***Bilinearity***) We showed in the previous remarks that $[,]_{\mathfrak{e}_8}$ is bilinear.

(***Anti-commutativity***) We consider several cases:

(i) $[a,b]_{\mathfrak{e}_8} = [a,b]_{\mathfrak{spin}(16)} = -[b,a]_{\mathfrak{spin}(16)} = -[b,a]_{\mathfrak{e}_8} \;\; \forall a,b \in \mathfrak{spin}(16)$.

(ii) $[a,u]_{\mathfrak{e}_8} \doteqdot -[u,a]_{\mathfrak{e}_8} \quad \forall a \in \mathfrak{spin}(16),\; u \in \triangle^+$.

(iii) $[u,v]_{\mathfrak{e}_8} = -\;[v,u]_{\mathfrak{e}_8}$. This is because $\forall a \in \mathfrak{spin}(16),\; u,v \in \triangle^+$, we have:

$\left\langle a, [u,v]_{\mathfrak{e}_8} + [v,u]_{\mathfrak{e}_8} \right\rangle = \left\langle a, [u,v]_{\mathfrak{e}_8} \right\rangle + \left\langle a, [v,u]_{\mathfrak{e}_8} \right\rangle = \left\langle [a,u]_{\mathfrak{e}_8}, v \right\rangle + \left\langle [a,v]_{\mathfrak{e}_8}, u \right\rangle \overset{\text{by symmetry}}{=} \left\langle [a,u]_{\mathfrak{e}_8}, v \right\rangle + \left\langle u, [a,v]_{\mathfrak{e}_8} \right\rangle = 0$, by invariance of $\langle , \rangle$ under $a \in \mathfrak{spin}(16)$.

Since this holds for any $a \in \mathfrak{spin}(16)$, we conclude that $[u,v]_{\mathfrak{e}_8} + [v,u]_{\mathfrak{e}_8} = 0$ and so $[u,v]_{\mathfrak{e}_8} = -[v,u]_{\mathfrak{e}_8}$.

(***Jacobi identity***) We consider several cases:

(i) All of the three variables are in $\mathfrak{spin}(16)$.
This would then be the Jacobi identity for $\mathfrak{spin}(16)$.

(ii) Two variables are in $\mathfrak{spin}(16)$ and the third is in $\triangle^+$.
In this case, we have: $[[a,b],u] = [a,[b,u]] - [b,[a,u]] \Longrightarrow [a,[b,u]] + [u,[a,b]] + [b,[u,a]] = 0$.

(iii) One variable is in $\mathfrak{spin}(16)$ and the other two are in $\triangle^+$.
By invariance of the bracket under $\mathfrak{spin}(16)$, we have that $[a,[u,v]] = [[a,u],v] + [u,[a,v]]$ which leads to the Jacobi identity. (Look at 3.8).

(iv) All of the three variables are in $\triangle^+$.
This is the case where we use the fact that we are working over $\mathfrak{spin}(16)$. The proof can be found in [1]. ■

(3.12) Let $\mathfrak{e}_8 = \mathfrak{spin}(16) + \triangle^+$ with the invariant inner product $\langle , \rangle_{\mathfrak{e}_8}$ and the Lie bracket $[,]_{\mathfrak{e}_8}$.

Then $\dim(\mathfrak{e}_8) = \dim(\mathfrak{spin}(16)) + \dim(\triangle^+) = \binom{16}{2} + 2^{8-1} = 120 + 128 = 248$, and it can be shown that the roots of $\mathfrak{e}_8$ are:

· $\pm x_i \pm x_j$, $1 \le i < j \le 8$, coming from $\mathfrak{spin}(16)$ and there are $4\binom{8}{2} = 112$ of them.

· $\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_8)$ (even number of $-$ signs) coming from $\triangle^+$ and there are $\displaystyle\sum_{n=0}^{4} \binom{8}{2n} = 128$ of them.

All these roots have the same length, $= 2$, and it can be shown that a base may be given by the simple roots:

1	$\frac{1}{2}(x_1 + x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8)$
2	$x_2 + x_3$
3	$-x_2 + x_3$
4	$-x_3 + x_4$
5	$-x_4 + x_5$
6	$-x_5 + x_6$
7	$-x_6 + x_7$
8	$-x_7 + x_8$

Look at the Appendix to see the roots as matrices.

Most of the calculations for the inner product were done in 3.2. The ones left are:

$\langle 1,1\rangle = 2$, $\langle 1,2\rangle = 0$, $\langle 1,3\rangle = \langle 3,1\rangle = -1$, $\langle 1,4\rangle = 0$, $\langle 1,5\rangle = 0$, $\langle 1,6\rangle = 0$, $\langle 1,7\rangle = 0$, $\langle 1,8\rangle = 0$.
Also, remember that $\langle 2,3\rangle = 0$. Moreover, $\langle i,i+1\rangle = \langle i+1,i\rangle = -1$ for $3 \le i \le 7$.
Finally, $\langle 2,4\rangle = \langle 4,2\rangle = -1$.

Thus, the Cartan Matrix is: $\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$,

and the Dynkin diagram is:

$$\begin{array}{ccccccccccccc} & & & & \overset{2}{\mathbf{O}} & & & & & & & & \\ & & & & | & & & & & & & & \\ \underset{1}{\mathbf{O}} & - & \underset{3}{\mathbf{O}} & - & \underset{4}{\mathbf{O}} & - & \underset{5}{\mathbf{O}} & - & \underset{6}{\mathbf{O}} & - & \underset{7}{\mathbf{O}} & - & \underset{8}{\mathbf{O}} \end{array}$$

Appendix A $\boxed{\text{The Roots of } \mathfrak{e}_8}$

Here are the roots as matrices: (Look at 3.12)

$$1 \qquad \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}$$

$$2 \qquad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$3 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$4 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$$
5 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
6 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
7 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
8 \qquad \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0
\end{pmatrix}
$$

Appendix B $\boxed{\text{The Exceptional Lie Group } \mathrm{E}_8}$

$$\boxed{\dim(E_8) = 248}$$

$$\boxed{\textit{Dynkin Diagram}} = \begin{array}{ccccccccccccc} & & & & \overset{2}{\mathbf{O}} & & & & & & & & \\ & & & & | & & & & & & & & \\ \underset{1}{\mathbf{O}} & - & \underset{3}{\mathbf{O}} & - & \underset{4}{\mathbf{O}} & - & \underset{5}{\mathbf{O}} & - & \underset{6}{\mathbf{O}} & - & \underset{7}{\mathbf{O}} & - & \underset{8}{\mathbf{O}} \end{array}$$

$$\boxed{\textit{Cartan Matrix of } E_8} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$\boxed{\textit{The positive roots of } E_8}$ *are:*

[1,0,0,0,0,0,0,0], [0,1,0,0,0,0,0,0], [0,0,1,0,0,0,0,0], [0,0,0,1,0,0,0,0],
[0,0,0,0,1,0,0,0], [0,0,0,0,0,1,0,0], [0,0,0,0,0,0,1,0], [0,0,0,0,0,0,0,1],
[1,0,1,0,0,0,0,0], [0,1,0,1,0,0,0,0], [0,0,1,1,0,0,0,0], [0,0,0,1,1,0,0,0],
[0,0,0,0,1,1,0,0], [0,0,0,0,0,1,1,0], [0,0,0,0,0,0,1,1], [1,0,1,1,0,0,0,0],
[0,1,1,1,0,0,0,0], [0,1,0,1,1,0,0,0], [0,0,1,1,1,0,0,0], [0,0,0,1,1,1,0,0],
[0,0,0,0,1,1,1,0], [0,0,0,0,0,1,1,1], [1,1,1,1,0,0,0,0], [1,0,1,1,1,0,0,0],
[0,1,1,1,1,0,0,0], [0,1,0,1,1,1,0,0], [0,0,1,1,1,1,0,0], [0,0,0,1,1,1,1,0],
[0,0,0,0,1,1,1,1], [1,1,1,1,1,0,0,0], [1,0,1,1,1,1,0,0], [0,1,1,2,1,0,0,0],
[0,1,1,1,1,1,0,0], [0,1,0,1,1,1,1,0], [0,0,1,1,1,1,1,0], [0,0,0,1,1,1,1,1],
[1,1,1,2,1,0,0,0], [1,1,1,1,1,1,0,0], [1,0,1,1,1,1,1,0], [0,1,1,2,1,1,0,0],
[0,1,1,1,1,1,1,0], [0,1,0,1,1,1,1,1], [0,0,1,1,1,1,1,1], [1,1,2,2,1,0,0,0],
[1,1,1,2,1,1,0,0], [1,1,1,1,1,1,1,0], [1,0,1,1,1,1,1,1], [0,1,1,2,2,1,0,0],
[0,1,1,2,1,1,1,0], [0,1,1,1,1,1,1,1], [1,1,2,2,1,1,0,0], [1,1,1,2,2,1,0,0],
[1,1,1,2,1,1,1,0], [1,1,1,1,1,1,1,1], [0,1,1,2,2,1,1,0], [0,1,1,2,1,1,1,1],
[1,1,2,2,2,1,0,0], [1,1,2,2,1,1,1,0], [1,1,1,2,2,1,1,0], [1,1,1,2,1,1,1,1],
[0,1,1,2,2,2,1,0], [0,1,1,2,2,1,1,1], [1,1,2,3,2,1,0,0], [1,1,2,2,2,1,1,0],
[1,1,2,2,1,1,1,1], [1,1,1,2,2,2,1,0], [1,1,1,2,2,1,1,1], [0,1,1,2,2,2,1,1],
[1,2,2,3,2,1,0,0], [1,1,2,3,2,1,1,0], [1,1,2,2,2,2,1,0], [1,1,2,2,2,1,1,1],
[1,1,1,2,2,2,1,1], [0,1,1,2,2,2,2,1], [1,2,2,3,2,1,1,0], [1,1,2,3,2,2,1,0],
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[2,3,4,6,5,3,2,1], [2,3,4,6,5,4,2,1], [2,3,4,6,5,4,3,1], [2,3,4,6,5,4,3,2]

$\boxed{\textit{The dimensions of the fundamental representations of } E_8}$ are:

[1,0,0,0,0,0,0,0]	3 875
[0,1,0,0,0,0,0,0]	147 250
[0,0,1,0,0,0,0,0]	6 696 000
[0,0,0,1,0,0,0,0]	6 899 079 264
[0,0,0,0,1,0,0,0]	146 325 270
[0,0,0,0,0,1,0,0]	2 450 240
[0,0,0,0,0,0,1,0]	30 380
[0,0,0,0,0,0,0,1]	248

$\boxed{\text{The nice result about the Lie algebra, } \mathfrak{e}_8\text{, of } E_8}$ is $\boxed{\mathfrak{e}_8 = \mathfrak{spin}(16) + \triangle^+}$

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