Hyperbolic Schrödinger Equation

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Abstract. Clifford algebra corresponds to Minkwoski space. The coupling between real object particles and light quantums can be discussed by Minkowski space's directional strangeness. We introduce Galilei transfomation and Schrödinger equation into Minkowski space and give a geometrical explanation for classical quantum theory.

1. The Correlation Between Minkowski Space and Galilei Transformation

In traditional relativity theory, when Lorentz transformation is classical approximation ($v << c$), there is the transition from it to Galilei transformation. In the meantime, we take the transition from Minkowski space to Euclidean space without any analysis. In real number field, the correspondence between Galilei transformation and three dimensional Euclid space is unquestionable. But, it deserves discussing in complex field.

In hyperbolic Minkowski space, we take two physical events $X_1(x_1, jct_1)$ and X_2 (x_2, jct_2). Here, $j(j^2 = 1, j \neq 1, j^* = -j)$ [1] is hyperbolic virtual unit. When $x_1 << ct_1$ and $x_2 << ct_2$, or $v << c$, the real object particles corresponded by the two events are at low speed and satisfy Galilei transformation. But, Minkowski space has directional strangeness character [2, 3]. If the connecting line between the space-time points X_1 and X_2 in time-like zone, is vertical or parallel to light-like zone Ξ, there is $X_1 - X_2 \in \Xi$. For example, taking $X_1 = 1 + j1000$ and $X_2 = 2 + j1001$, if we assume 1000>>1, we have $X_1 - X_2 = 1 + j \in \Xi_1$ [4]. So we can characterize the duality of microscopic low speed particles in Minkowski space. On the one hand, we can use Galilei transformation to characterize microscopic low speed particle's movement sep-

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arately. The corresponding space-time coordinates can be classified into a three dimensional space coordinates and an independent one dimensional time coordinates approximately. This is similar to traditional Euclid space's property. On the other hand, the coupling between real object particles and light quantums or the causation between low speed particles and light quantums is in relation to Minkowski space's directional strangeness, but not to Euclid space's property. The motive characteristic of light quantums satisfies Lorentz transformation. The representation of Microscopic real objects is wave-particle duality. So Minkowski space does not only correspond to Lorentz transformation but also to Galilei transformation. On that account, the restricted theory of relativity can characterize microscopic low speed particles' motive action in Minkowski space rather than being translated into Euclid space for classical approximation. The classical quantum mechanics can be introduced into non-Euclidean geometry by the correspondence between hyperbolic Minkowski space and Galilei transformation. This gives us a way to find the logic correlation between the classical quantum mechanics and the restricted theory of relativity.

2. Hyperbolic Fourier Transformation

Fourier transformation is the indispensable mathematical tool in the quantum mechanics. But we should build up Fourier transformation theory corresponded by it in hyperbolic Minkowski space, namely hyperbolic Fourier transformation. In hyperbolic Minkowski space, hyperbolic complex number can be written in hyperbolic function and power series form

$$X = x + jct = jR(ch\varphi + jsh\varphi) = jRe^{j\varphi} = jR\sum_{n=0}^{\infty}\frac{(j\varphi)^2}{n!} \tag{1}$$

For general hyperbolic complex function, referring to the traditional Fourier transformation theory, it can be expanded to hyperbolic Fourier transformation series in the interval $[-l, l]$,

$$\begin{aligned} f(X) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n ch\frac{n\pi X}{l} + b_n sh\frac{n\pi X}{l}) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{a_n}{2}(e^{\frac{jn\pi X}{l}} + e^{-\frac{jn\pi X}{l}}) + \sum_{n=1}^{\infty}\frac{b_n}{2j}(e^{\frac{jn\pi X}{l}} + e^{-\frac{jn\pi X}{l}}) \end{aligned} \tag{2}$$

Or

$$f(X)=\sum_{n=-\infty}^{+\infty}C_n e^{\frac{jn\pi X}{l}}(-l\le X\le l) \tag{3}$$

Here

$$C_0=\frac{a_0}{2},C_n=\frac{1}{2}(a_n+jb_n),C_{-n}=\frac{1}{2}(a_n-jb_n) \tag{4}$$

It satisfies

$$C_{-n}C_n=C_n^*C_n=\frac{1}{4}\left|a_n^2-b_n^2\right| \tag{5}$$

Let $k_n=\frac{n\pi}{l}$, then

$$f(X)=\sum_{n=-\infty}^{+\infty}C_n e^{jk_nX} \tag{6}$$

For (6), multiplied by e^{-jk_mX}, and integrated for the whole space

$$\int_{-\infty}^{+\infty}e^{-jk_mX}f(X)dx=\sum_{-\infty}^{+\infty}C_n(k)\int_{-\infty}^{+\infty}e^{-jk_mX}e^{jk_nX}dx \tag{7}$$

Taking $\sum_{-\infty}^{+\infty}e^{jk_n}$ as a group of orthogonal radix base, we have

$$\int_{-\infty}^{+\infty}e^{j(k_n-k_m)X}dx=\delta(k_n-k_m) \tag{8}$$

Here $\delta(k_n-k_m)$ is Dirac δ function, then

$$\int_{-\infty}^{+\infty}e^{-jk_mX}f(X)dx=\sum_{-\infty}^{+\infty}C_n(k)\delta_{nm}=C_m(k) \tag{9}$$

Let $e^{jk_nX}=\psi_{n(X)}$, take it into (6), we have

$$f(X)=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi_n^*(X')f(X')\psi_n(X)dx'=\int_{-\infty}^{+\infty}f(X')\delta(X-X')dx' \tag{10}$$

(10) can be called hyperbolic Fourier series. Here

$$\delta(X - X') = \sum_{n=-\infty}^{+\infty} \psi_n^*(X')\psi_n(X) \tag{11}$$

Let $\Delta k = 1$, (6) can be written as

$$f(X) = \sum_{n=-\infty}^{+\infty} C_n(k)e^{jk_nX}\Delta k$$

Or

$$f(X) = \int_{-\infty}^{+\infty} C(k)e^{jkX}dk \tag{12}$$

Here

$$C(k) = \int_{-\infty}^{+\infty} f(X)e^{-jkX}dx \tag{13}$$

(12) and (13) are called hyperbolic Fourier integration.

3. The Mechanical Quantity Expressed by Operator

In hyperbolic Hilbert space [5], the average value of mechanical quantity is

$$\bar{A} = \int_{(\mu)} \psi^* A\psi d^{(\mu)}x$$

Here $\psi = \sum_n C_n\psi_n$ is universal set. $d^{(\mu)}x$ is μ dimensions volume element. The integral region is μ dimensions full space integration. In four dimensions hyperbolic quantum space, taking average quantum value

$$\bar{P}_\mu = \int_{-\infty \atop (\mu=4)}^{+\infty} \varphi^*(P_\mu)P_\mu\varphi(P_\mu)d^{(\mu)}p \tag{14}$$

By (13), taking

$$\varphi(P_\mu) = \int_{-\infty \atop (\mu=4)}^{+\infty} \psi(X_\mu)e^{-\frac{jP_\mu X_\mu^*}{\hbar}}d^{(\mu)}x \tag{15}$$

Taking it into (14), then

$$\bar{P}_\mu = \int_{\substack{-\infty \\ (3\mu)}}^{+\infty} \psi^*(X_\mu'^*) e^{\frac{jP_\mu^* X_\mu'}{\hbar}} P_\mu \psi(X_\mu) e^{\frac{-jP_\mu X_\mu^*}{\hbar}} d^{(\mu)}pd^{(\mu)}x'd^{(\mu)}x \tag{16}$$

For $P_\mu^* X_\mu = P_\mu X_\mu^*$, so

$$\bar{P}_\mu = \int_{\substack{-\infty \\ (2\mu)}}^{+\infty} \psi^*(X_\mu'^*) e^{\frac{jP_\mu^* X_\mu'}{\hbar}} d^{(\mu)}pd^{(\mu)}x' \int_{\substack{-\infty \\ (\mu)}}^{+\infty} \psi(X_\mu)(-j\hbar\Box^*) e^{\frac{-jP_\mu X_\mu^*}{\hbar}} d^{(\mu)}x \tag{17}$$

Here $\Box^*$ is four dimensions gradient operator [6]. We have

$$\begin{aligned} \Box &= \frac{\partial}{\partial X_\mu} = \nabla + j\frac{\partial}{\partial(ct)} \\ \Box^* &= \frac{\partial}{\partial X^\mu} = \nabla - j\frac{\partial}{\partial(ct)} \end{aligned} \tag{18}$$

Here wave function $\psi(X_\mu)$ must be single valued and continuous. It is zero in its boundary $(\pm\infty)$. Using integration by parts for the latter part of (17),

$$\bar{P}_\mu = \int_{\substack{-\infty \\ (2\mu)}}^{+\infty} \psi^*(X_\mu'^*) e^{\frac{jP_\mu X_\mu'^*}{\hbar}} d^{(\mu)}pd^{(\mu)}x' \int_{\substack{-\infty \\ (\mu)}}^{+\infty} e^{\frac{-jP_\mu X_\mu^*}{\hbar}} (j\hbar\Box^*)\psi(X_\mu) d^{(\mu)}x \tag{19}$$

By

$$f^*(X_\mu^*) = \int_{\substack{-\infty \\ (2\mu)}}^{+\infty} f^*(X_\mu'^*) e^{\frac{jP_\mu(X'^* - X^*)}{\hbar}} d^{(\mu)}pd^{(\mu)}x' \tag{20}$$

Taking (20) into (19), we have

$$\bar{P}_\mu = \int_{\substack{-\infty \\ (\mu)}}^{+\infty} \psi^*(X_\mu^*) \hat{P}_\mu \psi(X_\mu) d^{(\mu)}x \tag{21}$$

Here

$$\hat{P}_\mu = j\hbar\Box^* \tag{22}$$

(22) is four dimensions hyperbolic quantum operator and can be written in the form

$$\widehat{\vec{p}} = j\hbar\nabla, \widehat{H} = -j\hbar\frac{\partial}{\partial(ct)} \tag{23}$$

(23) is energy and quantum operator.

4. Hyperbolic Schrödinger Equation

To a hyperbolic Minkowski space corresponds a Galilei transformation. This makes it possible to discuss microscopic low speed foreign body's law of motion and Schrödinger equation. Taking hyperbolic wave function

$$\psi = e^{\frac{jX_\mu^* P_\mu}{\hbar}} = Ae^{\frac{j(\vec{p}\cdot\vec{r}-Et)}{\hbar}} \tag{24}$$

We have the intrinsic equation

$$\widehat{P}_\mu \psi = j\hbar \Box^* \psi = \widehat{\vec{p}}\psi + j\frac{\widehat{H}}{c}\psi = \vec{p}\psi + j\frac{E}{c}\psi = P_\mu \psi \tag{25}$$

Separately, it can be written in the form

$$\widehat{H}\psi = E\psi, \widehat{\vec{p}}\ \psi = \vec{p}\psi \tag{26}$$

For classical approximation, taking $c = 1$ and $m_0 = m$, expanding $E^2 = p^2c^2 + m_0^2c^4$ by $\frac{p}{m}$, and taking first class approximation, we obtain

$$E = m + \frac{p^2}{2m} \tag{27}$$

Taking (27) into (24) crnd dividing ψ by $e^{\frac{-jmt}{\hbar}}$, we have

$$\phi = \psi e^{\frac{jmt}{\hbar}} = e^{\frac{j}{\hbar}(\vec{p}\ \vec{r} - \frac{p^2}{2m}t)} \tag{28}$$

Taking differential quotient for time

$$\frac{\partial \phi}{\partial t} = -\frac{jp^2}{2\hbar m}\phi \tag{29}$$

By (29) and (26), we have

$$\frac{\partial \phi}{\partial t} + j\frac{\hbar}{2m}\nabla^2\phi = 0 \tag{30}$$

(30) is hyperbolic Schrödinger equation. Taking Hermitian conjugate for (30), we have

$$\frac{\partial \phi^+}{\partial t} - j\frac{\hbar}{2m}\nabla^2\phi^+ = 0 \tag{31}$$

Adding (30) multiplied by ϕ^+ on the left, to (31) multiplied by ϕ on the right, we get

$$\Box J_\mu = \frac{\partial J_\mu}{\partial X_\mu} = \frac{\partial \rho}{\partial t} + \nabla \vec{J} = 0 \tag{32}$$

Here

$$J_\mu = \vec{J} + j\rho \tag{33}$$

Or

$$\begin{aligned} \rho &= \phi^+\phi \\ \vec{J} &= \tfrac{j\hbar}{2m}(\phi^+\nabla\phi - \nabla\phi^+\phi) \end{aligned} \tag{34}$$

They are probability density and probability current density [6, 7] that the hyperbolic Schrödinger equation corresponds with.

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