CLIFFORD ALGEBRA AND QUANTUM LOGIC GATES

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(Received: August 23, 2004; Accepted: October 29, 2004)

Abstract. This paper discusses the properties of Quantum bit (Qubit) and Quantum logic gates (Quantum not-gate, Hadamard gate and Quantum controlled not-gate etc.) by the generating element of Pauli algebra (Clifford algebra Cl_3).

Key words: Clifford algebra; quantum logic gates; quantum bit; Pauli matrix

1. Clifford Algebra Cl_3 and its Matrix Representation

The generating space R^3 of Clifford algebra Cl_3 is a three-dimensional real linear space and its basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ under the Clifford product of Cl_3 satisfies [1]:

$$\mathbf{e}_i\mathbf{e}_j = \begin{cases} 1, & if \quad i = j, \\ -\mathbf{e}_j\mathbf{e}_i, & if \quad i \neq j. \end{cases} \tag{1.1}$$

Cl_3 is $2^3 = 8$ dimensional real associative algebra and it has a basis: $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}$, where $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j, \mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

Take Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.2}$$

Advances in Applied Clifford Algebras **14** No. 2, 225-230 (2004)

Let $\sigma_{ij} = \sigma_i\sigma_j, \sigma_{123} = \sigma_1\sigma_2\sigma_3, \sigma_0 = I_2$ then

$$\begin{aligned} G_3 = \ & \{a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_{12}\sigma_{12} + a_{13}\sigma_{13} + a_{23}\sigma_{23} + a_{123}\sigma_{123} \\ & \quad |a_x \in R, x \in \{0, 1, 2, 3, 12, 13, 23, 123\}\} \end{aligned} \tag{1.3}$$

is an 8-dimensional real associative algebra and is called Pauli algebra.

In fact, $G_3 = \mathrm{Mat}(2, C) \simeq Cl_3(f : Cl_3 \longrightarrow \mathrm{Mat}(2, C), \mathbf{e}_i \to \sigma_i, i \in \{1, 2, 3\}$ is an algebraic isomorphic mapping). We will discuss the properties of quantum logic gates by the generating element of Pauli algebra $\sigma_i (i \in \{1, 2, 3\})$ and σ_0.

2. Quantum Bit and Quantum Logic Gates

The state space of a single quantum bit (qubit) is a two-dimensional Hilbert space H_2 and its basis states are represented by Dirac notation $|0\rangle, |1\rangle$, whose conjugates are denoted by $\langle 0|, \langle 1|$, the corresponding matrix forms of $|0\rangle, |1\rangle$ and $\langle 0|, \langle 1|$ are [2-4]

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; [1 \ \ 0], [0 \ \ 1]. \tag{2.1}$$

Any operation of n-qubit logic gates can be finally expressed as the operation of 1-qubit and 2-qubit logic gates [5], pay close attention to this, firstly, we give some matrix representation of the typical 1-qubit and 2-qubit logic gates, then give some matrix representation of n-qubit logic gates.

Example 1. By (1.2) and (2.1), we obtain

$$\sigma_1|0\rangle = |1\rangle, \ \sigma_1|1\rangle = |0\rangle \tag{2.2}$$

So Pauli matrix σ_1 is called **quantum not-gate** and denoted $\sigma_1 = Q_N$.

Example 2. Let $Q_H = \sqrt{\frac{1}{2}}(\sigma_1 + \sigma_3)$, then

$$Q_H|0\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle), Q_H|1\rangle = \sqrt{\frac{1}{2}}(|0\rangle - |1\rangle). \tag{2.3}$$

Q_H is called **Hadamard gate** or **Walsh gate**.

Quantum not-gate Q_N and Hadamard gate Q_H satisfy

$$Q_N^2 = Q_H^2 = I_2 = \sigma_0. \tag{2.4}$$

Example 3 (phase gate). The phase gate $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ can be represented by Pauli matrix as

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \frac{1}{2}(\sigma_0 + \sigma_3) + \frac{i}{2}(\sigma_0 - \sigma_3) = \frac{1+i}{2}\sigma_0 + \frac{1-i}{2}\sigma_3. \tag{2.5}$$

Example 4 ($\frac{\pi}{8}$ gate). $\frac{\pi}{8}$ gate or called T gate $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$ can be represented by Pauli matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \frac{1+e^{\frac{i\pi}{4}}}{2}(\sigma_0+\sigma_3)+\frac{i}{2}(\sigma_0-\sigma_3) = \frac{1+e^{\frac{i\pi}{4}}}{2}\sigma_0+\frac{1-e^{\frac{i\pi}{4}}}{2}\sigma_3. \tag{2.6}$$

Example 5 (rotation gate). The rotation gate $R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$ can be represented by Pauli matrix as

$$R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \frac{1+e^{i\theta}}{2}\sigma_0 + \frac{1-e^{i\theta}}{2}\sigma_3. \tag{2.7}$$

the phase gate $S = R_{\frac{\pi}{2}}$ and $\frac{\pi}{8}$ gate $T = R_{\frac{\pi}{4}}$.

The state space of a two-qubit is a four-dimensional Hilbert space $H_2^2 = H_2 \otimes H_2$. The basis states of H_2^2 denoted by $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and the corresponding matrix forms are

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, |01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$|10\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, |11\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.8}$$

Example 6. Define

$$Q_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.9}$$

is called quantum **controlled not-gate**, and

$$Q_{CN}|00\rangle = |00\rangle, Q_{CN}|01\rangle = |01\rangle, Q_{CN}|10\rangle = |11\rangle, Q_{CN}|11\rangle = |10\rangle. \tag{2.10}$$

Obviously, $Q_{CN}^2 = I_4$, and is represented by Pauli matrix as

$$Q_{CN} = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{bmatrix}. \tag{2.11}$$

Ordinarily, the state space of an n-qubit is a 2^n dimensional Hilbert space

$$H_2^n = H_2 \otimes H_2 \otimes \cdots \otimes H_2 = H_2^{\otimes n}. \tag{2.12}$$

The basis states of H_2^n are represented by

$$|\delta_1\delta_2 \cdots \delta_n\rangle \equiv |\delta_1\rangle|\delta_2\rangle \cdots |\delta_n\rangle \equiv |\delta_1\rangle \otimes |\delta_2\rangle \otimes \cdots \otimes |\delta_n\rangle, \tag{2.13}$$

where $\delta_i \in \{0, 1\}, i \in \{1, 2, \cdots, n\}$.

Example 7. Let $W_2 \equiv Q_N$, define quantum logic gate of 2-qubit W_4 as follows

$$W_4 = W_2 \otimes W_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{2.14}$$

W_4 can be represented by Pauli matrix as

$$W_4 = \frac{1}{2} \begin{bmatrix} \sigma_1 + \sigma_3 & \sigma_1 + \sigma_3 \\ \sigma_1 + \sigma_3 & -(\sigma_1 + \sigma_3) \end{bmatrix}, \tag{2.15}$$

and

$$\begin{aligned} W_4|x_0x_1\rangle &= \tfrac{1}{2}(|0\rangle + (-1)^{x_0}|1\rangle)(|0\rangle + (-1)^{x_1}|1\rangle) \\ &= \tfrac{1}{2}(|00\rangle + (-1)^{x_1}|01\rangle + (-1)^{x_0}|10\rangle + (-1)^{x_0+x_1}|11\rangle). \end{aligned} \tag{2.16}$$

Define a quantum logic gate of n-qubit W_{2^n} as follows

$$W_{2^n} = W_2 \otimes W_{2^{n-1}} = \begin{bmatrix} W_{2^{n-1}} & W_{2^{n-1}} \\ W_{2^{n-1}} & -W_{2^{n-1}} \end{bmatrix}, \tag{2.17}$$

is called n-qubit Hadamard gate.

Example 8. Let

$$Q_T = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}, \tag{2.18}$$

is called Toffoli gate and it can be represented by Pauli matrix as

$$Q_T = \begin{bmatrix} \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_0 & 0 & 0 \\ 0 & 0 & \sigma_0 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{bmatrix} = \begin{bmatrix} \sigma_0^{\otimes 2} & 0 \\ 0 & T_2 \end{bmatrix}, \tag{2.19}$$

where $T_2 = Q_{CN}$ and

$$Q_T|x_0x_1x_2\rangle = |x_0x_1(x_2 \oplus x_0x_1)\rangle, \tag{2.20}$$

where $0 \oplus 0 = 1 \oplus 1 = 0, 0 \oplus 1 = 1 \oplus 0 = 1$.

Let

$$T_{2^n} = \begin{bmatrix} I_{2^{n-1}} & 0 \\ 0 & T_{2^{n-1}} \end{bmatrix} \tag{2.21}$$

is called n-qubit XOR gate and it can be represented by σ_0 as

$$T_{2^n} = \begin{bmatrix} \sigma_0^{\otimes n-1} & 0 \\ 0 & T_{2^{n-1}} \end{bmatrix}. \tag{2.22}$$

When $n = 2$, the XOR gate is quantum controlled not-gate Q_{CN}, when $n = 3$, the XOR gate is Toffoli gate Q_T, and

$$T_{2^n}|x_0x_1 \cdots x_{n-2}x_{n-1}\rangle = |x_0x_1 \cdots x_{n-2}(x_{n-1} \oplus x_0x_1 \cdots x_{n-2})\rangle. \tag{2.23}$$

Example 9 (Bell basis). In the state space of 2-qubit the $H_2^2 = H_2 \otimes H_2$, take orthonormal basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Let

$$\begin{aligned} |\Phi^{\pm}\rangle &= \sqrt{\tfrac{1}{2}}(|00\rangle \pm |11\rangle), \\ |\Psi^{\pm}\rangle &= \sqrt{\tfrac{1}{2}}(|01\rangle \pm |10\rangle). \end{aligned} \tag{2.24}$$

The states in (2.24) also form an orthonormal basis of H_2^2, and is called Bell basis.

By Pauli matrices to form quantum logic gates of 2-qubit as follows

$$U_a = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_3 \end{bmatrix}, U_b = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, U_c = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, U_d = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1\sigma_3 \end{bmatrix}. \tag{2.25}$$

By (2.24) and (2.25), we obtain

$$\begin{aligned} &U_a|\Phi^{\pm}\rangle = |\Phi^{\mp}\rangle, U_b|\Psi^{\pm}\rangle = |\Psi^{\mp}\rangle; \\ &U_c|\Phi^{\pm}\rangle = |\Psi^{\pm}\rangle, U_d|\Phi^{\pm}\rangle = |\Psi^{\mp}\rangle. \end{aligned} \tag{2.26}$$

References

[1] Baylis W. E. “Clifford (Geometric) Algebra with Applications to Physics, mathematics, and engineering”, Birkh*ä*user, Boston, 1996.

[2] DiVincenzo D. P., Quantum computation, *Science*, **270** 255, (1995).

[3] DiVincenzo D. P., Two-bit gate are universal for quantum computation, *Phys. Rev. A*, **51** (2), 1015-1022 (1995).

[4] DiVincenzo D. P., Quantum gate and circuits, *Proc. R. Soc. London A*, **454** 261-276, (1998).

[5] Nielson Micael A. and Issac L. Chuang, “Quantum Computation and Quantum Information”, Cambridge University Press 2000.