NEW CONSTRUCTION OF CAUSAL TENSORS

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Abstract. We study the problem of constructing tensors satisfying the dominant property, a generalization of the dominant energy condition $T_{ab}u^a v^b \geq 0$ for all future directed causal vectors u, v. The construction is done on the paravector subspace of the r-fold Euclidean Clifford algebra $\bigotimes^r \mathcal{C}\ell_p$ and is a generalization of the representation of superenergy tensors with complex 2-spinors. Especially, as with 2-spinors, we are able to construct causal tensors of arbitrary rank, contrary to earlier constructions using tensors or the r-fold Lorentzian Clifford algebra $\bigotimes^r \mathcal{C}\ell_{p,1}$ that only produce causal tensors of even rank. An advantage of the construction in $\bigotimes^r \mathcal{C}\ell_p$ is that several algebraic properties become trivial due to the Euclidean norm on it.

1. Introduction

Tensors satisfying some positivity property like for example the dominant energy condition (DEC) $T_{ab}u^a v^b \geq 0$ for all future directed causal vectors u, v, are of great interest in general relativity and Lorentzian geometry. In relativity this is due to that the energy momentum tensors of physically realistic spacetimes should satisfy the DEC or some other energy condition. For example in the singularity theorems of Hawking and Penrose [9] the energy conditions act as constraints on the geometry, making it possible to prove global properties of the spacetime manifold and deduce the existence of singularities.

Generally in theories based on Lorentzian manifolds it is also useful to know whether a tensor satisfies some similar condition, for example the dominant property (DP)

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$$T_{a_1...a_r} u_1^{a_1} ... u_r^{a_r} \geq 0 \text{ for all future directed causal vectors } u_i^a \qquad (1)$$

which is a generalization of the DEC. A tensor that satisfies (1) is called a *causal* tensor. Given any tensor $S_{ab...c}$ Senovilla presented in [16] and [17] a method to construct the *superenergy* tensor $T\{S\}$ of S, that satisfies the dominant property above. Some special cases of this construction are the Bel-Robinson tensor [2]

$$T\{C\}_{abcd} = C_{akcl} C_b{}^k{}_d{}^l + {}^*C_{akcl} {}^*C_b{}^k{}_d{}^l$$

which is the superenergy of the conformal Weyl tensor C_{abcd} and the energy-momentum tensor

$$T\{F\}_{ab} = -\frac{1}{2}(F_{ac} F_b{}^c + {}^*F_{ac} {}^*F_b{}^c) = -F_{ac} F_b{}^c + \frac{1}{4} g_{ab} F_{cd} F^{cd}$$

of the electro magnetic field F_{ab}. Such tensors have found several applications like in [7] where the Bel-Robinson tensor was used in the study of the global nonlinear stability of the Minkowski spacetime. Other applications are on causal propagation of fields in [6], algebraic Rainich conditions in [5] and on the causal structure of Lorentzian manifolds in [8]. For a good description of applications one may consult [18].

Since Senovilla's definition of the superenergy tensors in [16],[17] there have been generalizations of the $s-e$ tensors using different formalisms. The first proof of the DP for the superenergy tensors in 4 dimensions defined in [16] was given by Bergqvist in [4] using 2-component spinors. In fact this approach was a strict generalization of the superenergy tensors in 4 dimensions. A proof of the DP in general dimension was given in [17]. Another generalization is presented by Pozo and Parra in [14] and [15] (see also [13]) using Clifford algebras of a Lorentzian vector space. It turns out that the Clifford algebraic formulation is very simple and natural. Moreover the proofs of the properties of the $s-e$ tensors become very elegant. However the spinorial construction is not naturally contained in the Clifford algebraic one. This can be seen from the fact that in the tensorial and Clifford algebraic constructions one always obtains an $s-e$ tensor of even rank, while it is natural, both from a mathematical and physical point of view, to have $s-e$ tensors of odd rank in the spinorial construction.

In this article we will present a construction of causal tensors in the r-fold Clifford algebra $\bigotimes^r \mathcal{C}\ell_p$, which is a generalization of the representation with

2-spinors. The main algebraic properties become almost trivial due to that we have an Euclidean norm on $\bigotimes^r \mathcal{C}\ell_p$.

We will use the following notation for the different operations in $\mathcal{C}\ell_{p,q}$. The main involution, is denoted by a hat $\widehat{A}$, the reverse of a multivector is denoted by a tilde $\widetilde{A}$ and for the Clifford conjugate we use an overbar $\overline{A}$. The p:th power of a vector space $\mathcal{V}$ is denoted by $\Lambda_p(\mathcal{V})$ or just Λ_p and the projection of a multivector A on this subspace is denoted by $\langle A\rangle_p$. Furthermore we use the notation $\langle A\rangle_{p\oplus q}$ for projection on the direct sum $\Lambda_p \oplus \Lambda_q$.

2. Paravectors and r-Fold Algebras

2.1. PARAVECTORS IN $\mathcal{C}\ell_p$

To be able to study Lorentz geometry within $\mathcal{C}\ell_p$ we will have to use a special subspace of $\mathcal{C}\ell_p$,

Definition 2.1 *The space* $\mathcal{P} = \Lambda_0 \oplus \Lambda_1 \subset \mathcal{C}\ell_p$ *is the space of paravectors.*

We denote paravectors with capital letters and see that a given vector $u = u_0e_0+u_1e_1+...+u_pe_p \in \mathbb{R}^{1,p}$ can be mapped to $U = u_0+u_1e_1+\ldots+u_pe_p \in \mathcal{P}$, if we identify the scalarpart of a paravector as the timelike component. Then by endowing $\mathcal{P}$ with the metric

$$g(U,V) = \langle U\overline{V}\rangle_0 = \langle V\overline{U}\rangle_0$$

we get a Lorentz vector space, since setting $V = v_0 + v_1e_1 + \ldots + v_pe_p$ and $U = u_0 + u_1e_1 + \ldots + u_pe_p$ gives

$$\begin{aligned}\langle U\overline{V}\rangle_0 &= \langle (u_0 + u_1e_1 + \ldots + u_pe_p)(v_0 - v_1e_1 - \ldots - v_pe_p)\rangle_0 \\ &= u_0v_0 - u_1v_1 - \ldots - u_pv_p\end{aligned}$$

When considering causal tensors the idea will be to construct these on $\mathcal{P}$. A given tensor $T \in \bigotimes^r \mathbb{R}^{1,p}$ can be mapped to $\bigotimes^r \mathcal{P}$ by sending timelike components to scalar components and the action of T on r vectors is given by

$$T(u_1,\ldots,u_r) = \langle T\overline{\underline{U}}\rangle_0$$

Here $\underline{U} = U_1\otimes\cdots\otimes U_r$ and $U_i \in \mathcal{P}$ is the paravector corresponding to the *vector* $u_i \in \mathbb{R}^{1,p}$. Transformations of the basis is done with Lorentz transformations. Using the isomorphism between the even subalgebra $\mathcal{C}\ell^+_{1,p}$ and $\mathcal{C}\ell_p$ one obtains the isomorphisms of groups $\$\Gamma_{1,p} \simeq \Gamma^+_{1,p}$ and $\$pin_{1,p} \simeq Spin_{1,p}$ [11], where

$$\$\Gamma_{1,p} = \{R \in \mathcal{C}\ell_p : \forall U \in \mathcal{P}, RU\widehat{R}^{-1} \in \mathcal{P}\}$$

and

$$\$pin_{1,p} = \{R \in \mathcal{C}\ell_p : \forall U \in \mathcal{P}, RU\widetilde{R} \in \mathcal{P} \text{ and } R\overline{R} = \pm 1\}$$

So we can use these groups for conformal and Lorentz transformations.

2.2. The r-fold algebra

Just as we can form tensor products of vector spaces we can form tensor products of algebras. We will study the algebra

$$\underbrace{\mathcal{C}\ell_{p,q} \otimes \cdots \otimes Cl_{p,q}}_{r} \equiv \bigotimes^{r} \mathcal{C}\ell_{p,q}$$

called the r-fold Clifford algebra with its elements being called r-fold multivectors. Factors in the tensor product $\bigotimes^r \mathcal{C}\ell_{p,q}$ will be called blocks. The notation can become quite cumbersome without the introduction of the so called multiindices, so we start by introducing them. Any m-vector A in $\mathcal{C}\ell_{p,q}$ may be written as

$$A = \sum_{1 \leq a_1 < \ldots < a_m \leq p+q} A^{a_1 \ldots a_m} e_{a_1} \wedge \ldots \wedge e_{a_m}$$

Note that the sum is over a given ordering of indices and no combination is repeated. In fact in counting the number of elements in this sum we get $\binom{p+q}{m}$, which is the dimension of Λ^m. If we then introduce the multiindices $I = (a_1 \ldots a_m)$ where the a_i run over such combinations, we can write A as

$$A = A^I e_I \text{ where } \{I\} = \{(a_1 \ldots a_m) : 1 \leq a_1 < \ldots < a_m \leq p+q\}$$

To study $\bigotimes^r \mathcal{C}\ell_{p,q}$, we can generalize this and introduce the so called multifold multiindices $\underline{I} = (I_1 \ldots I_r)$, where each I_i is a multiindex. Then an r-fold multivector A may be written

$$A = A^{I_1 \ldots I_r} e_{I_1} \otimes \ldots \otimes e_{I_r} = A^{\underline{I}} e_{\underline{I}}$$

We easily generalize the different operations on multivectors to r-fold multivectors. Reversion is given by

$$\widetilde{A} = A^{I_1 \ldots I_r} \widetilde{e}_{I_1} \otimes \ldots \otimes \widetilde{e}_{I_r}$$

Similarly, we can extend grade involution and Clifford conjugation to the r-fold algebra

$$\widehat{A} = A^{I_1\ldots I_r}\widehat{e_{I_1}} \otimes \cdots \otimes \widehat{e_{I_r}}$$

$$\overline{A} = A^{I_1\ldots I_r}\overline{e_{I_1}} \otimes \cdots \otimes \overline{e_{I_r}}$$

The projection operator can also be generalized, for example the projection of A on the subspace $\Lambda^{i_1} \otimes \ldots \otimes \Lambda^{i_r}$ is given by

$$\langle A\rangle_{(i_i,\ldots,i_r)} = A^{I_1\ldots I_r}\langle e_{I_1}\rangle_{i_1} \otimes \ldots \otimes \langle e_{I_r}\rangle_{i_r}$$

If all the i_k's are equal, we write

$$\langle A\rangle_{p,\ldots,p} = \langle A\rangle_p$$

Multiplication in tensor products of algebras is defined blockwise, so that for $A, B \in \bigotimes^r \mathcal{C}\ell_{p,q}$ we have

$$AB = A^{I_1\ldots I_r}B^{J_1\ldots J_r}e_{I_1}e_{J_1} \otimes \cdots \otimes e_{I_r}e_{J_r}$$

If we instead of the above multiply an r-fold multivector A and an s-fold multivector B we get

$$AB = A^{I_1\ldots I_r}B^{J_1\ldots J_s}e_{I_1}e_{J_1} \otimes \cdots \otimes e_{I_r}e_{J_r} \otimes e_{J_{r+1}} \otimes \cdots \otimes e_{J_s}$$

For example $A = e_1$ in Cl_3 and $B = e_1 \otimes e_3$ in $Cl_3 \otimes Cl_3$ results in $AB = 1 \otimes e_3$

3. Causal Tensors in 4 Dimensions

We look at the problem of how the 2-spinor construction of superenergy tensors can be made using the Clifford algebra $\mathcal{C}\ell_3$. In the formalism used in the book by Penrose and Rindler [12], a tensor formed as $T_{a_1\ldots a_r} = \Psi_{A_1\ldots A_r}\overline{\Psi}_{A'_1\ldots A'_r}$ satisfies the dominant property (1). The 2-spinors are elements of a minimal left ideal of this algebra and we start by discussing this in some detail in order to construct tensors like the one above using Clifford algebras.

3.1. SPINORS IN $\mathcal{C}\ell_3$

The exposition will follow the one given in [1]. A minimal left ideal can be formed by finding a primitive idempotent f. In $\mathcal{C}\ell_3$ a primitive idempotent is given by $f = \frac{1}{2}(1 + e)$ where e is a unit vector in $\mathbb{R}^3$ and our spinor space is then $\mathcal{C}\ell_3 f$. The trivector $I = e_{123}$ commutes with all vectors of $\mathbb{R}^3$ and therefore with the whole algebra. Moreover it satisfies $I^2 = -1$. Thus the field $\mathbb{F} = \{1, I\}$ generated by 1 and I is isomorphic to $\mathbb{C}$. We also note that bivectors are dual to vectors and hence of the form $e_i I$ with $e_i \in \mathbb{R}^3$. Hence $\mathcal{C}\ell_3$ is a four dimensional algebra over $\mathbb{F}$ with basis $\{1, e_1, e_2, e_3\}$, where the e_i are a basis for $\mathbb{R}^3$.

We can choose a unit vector n orthogonal to e and form the two elements $\alpha_0 = f$ and $\alpha_1 = nf$. Any multivector in $\mathcal{C}\ell_3$ can then be written

$$A = a_0 + a_1 e + a_2 n + a_3 I n e$$

with $a_i \in \mathbb{F}$, therefore

$$Af = (a_0 + a_1) f + (a_2 + a_3 I) n f = (a_0 + a_1)\alpha_0 + (a_2 + a_3 I)\alpha_1$$

From this we can see that $\mathcal{C}\ell_3 f$ is a 2 dimensional complex vector space, when the field of scalars is $\mathbb{F}$. In analogy with the 2-spinor formalism of [12] we can form an orthonormal tetrad using α_0 and α_1. First note that

$$\alpha_0 \widetilde{\alpha}_0 = f, \alpha_1 \widetilde{\alpha}_1 = \overline{f}, \alpha_0 \widetilde{\alpha}_1 = fn, \alpha_1 \widetilde{\alpha}_0 = nf$$

so that a tetrad $\{e_0, e_1, e_2, e_3\}$ spanning the paravector space can be formed as

$$\begin{aligned} e_0 &= 1 &&= \alpha_0 \widetilde{\alpha}_0 + \alpha_1 \widetilde{\alpha}_1 \\ e_3 &= e &&= \alpha_0 \widetilde{\alpha}_0 - \alpha_1 \widetilde{\alpha}_1 \\ e_1 &= n &&= \alpha_0 \widetilde{\alpha}_1 + \alpha_1 \widetilde{\alpha}_0 \\ I e_2 &= en &&= \alpha_0 \widetilde{\alpha}_1 - \alpha_1 \widetilde{\alpha}_0 \end{aligned}$$

Next let us see how null vectors in $\mathbb{R}^{1,3}$ can be represented with these spinors. A given spinor $\Psi \in \mathcal{C}\ell_3 f$ can be written as

$$\Psi = a_0 \alpha_0 + a_1 \alpha_1$$

Clifford conjugating this expression returns

$$\overline{\Psi} = a_0 \overline{\alpha}_0 + a_1 \overline{\alpha}_1$$

where $\overline{\alpha}_0 = \frac{1}{2}(1 - e)$ and $\overline{\alpha}_1 = -\frac{1}{2}(1 - e)n$. Note that $\overline{\alpha}_i \alpha_i = 0$ giving

$$\overline{\Psi}\Psi = a_0 a_1 \overline{\alpha}_0 \alpha_1 + a_0 a_1 \overline{\alpha}_1 \alpha_0 = 0$$

Next we note that order matters (which is natural since we have chosen a left ideal). To start with we have $\Psi\overline{\Phi} = 0$ for all $\Psi, \Phi \in \mathcal{C}\ell_3 f$ since we can write these as $\Psi = Af$ and $\overline{\Phi} = \overline{f}\ \overline{B}$ so that $\Psi\overline{\Phi} = Af\overline{f}\ \overline{B} = 0$ since $f\overline{f} = 0$. However $\overline{\Psi}\Phi$ is not zero unless Φ is a multiple of Ψ. The latter can be seen by setting $\Phi = b_0\alpha_0 + b_1\alpha_1$ and computing

$$\overline{\Psi}\Phi = a_0 b_1 \overline{\alpha}_0 \alpha_1 + b_0 a_1 \overline{\alpha}_1 \alpha_0$$

which returns zero iff $a_0/a_1 = b_0/b_1$. We can summarize all the above in

Proposition 3.1 *Let $T = \Psi\widetilde{\Psi}$, then T is a future directed null paravector with respect to the metric $< T\overline{T} >_0$ on the space of paravectors.*

Proof. Note that $\widetilde{T} = \widetilde{\Psi\widetilde{\Psi}} = \Psi\widetilde{\Psi} = T$ so that T must be a paravector. That it is null can be seen from $\langle T\overline{T}\rangle_0 = \langle \Psi\widetilde{\Psi}\overline{\Psi\widetilde{\Psi}}\rangle_0 = \langle \overline{\Psi}\Psi\widetilde{\Psi}\overline{\widetilde{\Psi}}\rangle_0 = 0$. Writing Ψ in the basis $\{\alpha_0, \alpha_1\}$ gives the expression, $(a_0\alpha_0 + a_1\alpha_1)(a_0^*\widetilde{\alpha}_0 + a_1^*\widetilde{\alpha}_1)$, where the star denotes complex conjugation (reversion changes the sign of I). The timelike component is given by $\alpha_0\widetilde{\alpha}_0 + \alpha_1\widetilde{\alpha}_1$ and has coefficient $a_0 a_0^* + a_1 a_1^*$ which is positive, hence T is future directed. ☐

Since we get future directed vectors we know that for $S = \Phi\widetilde{\Phi}$ and T as above, we get $\langle S\overline{T}\rangle_0 \geq 0$, with equality iff S and T are collinear. This can also be proved very elegantly with Clifford algebra methods. The following lemmas (see [10]) are almost trivial but very important

Lemma 3.2 *For $A \in \mathcal{C}\ell_3$, $\langle A\widetilde{A}\rangle_0 \geq 0$ and we have equality iff $A = 0$.*

Lemma 3.3 *Projection on the scalars obeys the following cyclic permutation property $\langle AB...CD\rangle_0 = \langle DAB...C\rangle_0$*

Proposition 3.4 *For any spinors $\Psi, \Phi \in \mathcal{C}\ell_3 f$ let $T = \Psi\widetilde{\Psi}$ and $S = \Phi\widetilde{\Phi}$ then $\langle T\overline{S}\rangle_0 \geq 0$ with equality iff T and S are collinear.*

Proof. Using the lemmas we have $\langle T\overline{S}\rangle_0 = \langle \Psi\widetilde{\Psi}\overline{\Phi\widetilde{\Phi}}\rangle_0 = \langle \overline{\Phi}\Psi\widetilde{\Psi}\overline{\widetilde{\Phi}}\rangle_0 = \langle \overline{\Phi}\Psi\widetilde{\overline{\Phi}\Psi}\rangle_0$ ≥ 0. We have equality iff $\overline{\Phi}\Psi = 0$ which by the above is equivalent to that Φ and Ψ are multiples of each other and generate collinear null paravectors. ☐

3.2. CAUSAL TENSORS FROM SPINORS IN $\bigotimes^r \mathcal{C}\ell_3 f$

Until now we have found that a spinor $\Psi \in \mathcal{C}\ell_3 f$ can be used to generate a causal paravector as $U = \Psi\widetilde{\Psi}$. It would then seem natural guess that for any $\Psi \in \bigotimes_{\mathbb{F}}^r \mathcal{C}\ell_3 f$, the tensor $T = \Psi\widetilde{\Psi}$ is causal. We make the following

Definition 3.5 *Let* $\Psi \in \bigotimes_{\mathbb{F}}^r \mathcal{C}\ell_3 f$ *and define* $T\{\Psi\}$ *by its action on a set of* $f-d$ *causal vectors* u_i *by* $T(u_1,\ldots,u_n) = \langle \widetilde{\Psi}\overline{\underline{U}}\Psi\,\rangle_0$, *where* $\underline{U}$ *is the* r*-fold paravector corresponding to* $u_1 \otimes \cdots \otimes u_r$.

That $T\{\Psi\}$ really satisfies the dominant property will be left to the next section, where we prove a more general result. Note that in this definition we consider $\mathcal{C}\ell_3 f$ as a vector space over $\mathbb{F} = \{1, I\}$ and therefore the tensor product satisfies $A \otimes_{\mathbb{F}} aB = aA \otimes_{\mathbb{F}} B$ where $a \in \mathbb{F}$. This is important as can be seen from the following example.

Example 3.6 *(The metric tensor): The metric is a trivial example of a causal tensor. In the usual 2-spinor formalism [12] we can construct the metric tensor as* $g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$. *The* ϵ_{AB} *is constructed from a spin basis* o_A, ι_A *as*

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B$$

In the left ideal $\mathcal{C}\ell_3 f$ *we can use the basis* $\{\alpha_0, \alpha_1\}$ *to construct* $\epsilon = \alpha_0 \otimes \alpha_1 - \alpha_1 \otimes \alpha_0$. *The metric then becomes*

$$g = \epsilon\widetilde{\epsilon} = \alpha_0\widetilde{\alpha}_0 \otimes \alpha_1\widetilde{\alpha}_1 + \alpha_1\widetilde{\alpha}_1 \otimes \alpha_0\widetilde{\alpha}_0 - \alpha_1\widetilde{\alpha}_0 \otimes \alpha_0\widetilde{\alpha}_1 - \alpha_0\widetilde{\alpha}_1 \otimes \alpha_1\widetilde{\alpha}_0$$

using the relations from section 3.1 we get

$$\begin{aligned} 2\epsilon\widetilde{\epsilon} &= 2f \otimes \overline{f} + 2\overline{f} \otimes f - 2nf \otimes fn - 2fn \otimes nf \\ &= 1 \otimes 1 - e \otimes e - n \otimes n + en \otimes en \\ &= e_0 \otimes e_0 - e_1 \otimes e_1 - e_3 \otimes e_3 + Ie_2 \otimes Ie_2 \end{aligned}$$

Now if we took $\mathcal{C}\ell_3 f$ *to be a real vector space we could not cancel the* I*'s from the last term. Thus we would not get a 2-fold paravector. But only the paravector part of* g *matters when we act on vectors so that the result would clearly not be that of the Minkowski metric. However since we have taken* $\mathcal{C}\ell_3 f$ *to be a vector space over* $\mathbb{F}$ *we can cancel the pseudoscalars from the last term and recover the Minkowski metric. Indeed being able to cancel* I *from any term assures that* $T\{\Psi\} = \Psi\widetilde{\Psi}$ *is always a tensor.*

We see that unless we take $\mathcal{C}\ell_3 f$ to be a complex vector space, we do not necessarily get multifold paravectors. The important point however is that we can construct superenergy tensors by using spinors in $\mathcal{C}\ell_3$ in a manner similar to the approach in [4]. We must also point out that if we want to generate all superenergy tensors in 4 dimensions, constructed from tensors, one must also use right ideals to take account for spinors of mixed types like for example $\Psi_{AB'}$. However we do not pursue this construction any further since it mainly serves as a starting point for a more general one.

4. Generalization to Arbitrary Dimension

4.1. DEFINITION AND ALGEBRAIC PROPERTIES

A natural way to generalize the above approach to causal tensors in $\mathcal{C}\ell_3 f$ is to consider the whole of $\mathcal{C}\ell_3$ instead. This can be made a 4-dimensional complex vector space and the approach is similar. However, in order to generalize this to other dimensions, we must abandon the complex structure. Instead we consider the real r-fold Clifford algebra $\bigotimes^r \mathcal{C}\ell_p$. The general definition is inspired by the earlier one with spinors,

Definition 4.1 *Let* $A \in \bigotimes^r \mathcal{C}\ell_p$ *and define the superenergy tensor* $T\{A\}$ *of* A *by its action on* r *vectors* $u_1, ..., u_r$ *according to* $T\{A\}(u_1, ..., u_r) = \langle \widetilde{A}\overline{\underline{U}}A\,\rangle_0$ *where* $\underline{U} = U_1 \otimes \cdots \otimes U_r$ *and* $U_i \in \mathcal{P}$ *is the paravector corresponding to the vector* u_i

Observe that one could make an equivalent definition by observing that only the paravector part of $A\widetilde{A}$ affects the outcome of $T\{A\}(u_1, ..., u_r)$. Thus we could also define the superenergy as the tensor $T\{A\} = \langle A\widetilde{A}\rangle_{0\oplus 1}$ belonging to the r-fold paravector space $\bigotimes^r \mathcal{P}$. The action on a set of r vectors would then be given by $T\{A\}(u_1, ..., u_r) = \langle T\{A\}\overline{\underline{U}}\,\rangle_0$. Now let us turn to the properties of $T\{A\}$. In order to prove the DP we need a few preliminary lemmas

Lemma 4.2 *For any* $A \in \bigotimes^r \mathcal{C}\ell_p$ *we have* $\langle A\widetilde{A}\rangle_0 \geq 0$, *with equality iff* $A = 0$.

Proof. We use multifold multiindices and write $A = A^{\underline{K}} e_{\underline{K}}$, then $A\widetilde{A} = (A^{\underline{K}} e_{\underline{K}})(A^{\underline{J}} \widetilde{e}_{\underline{J}})$. We can only get scalars when $\underline{K} = \underline{J}$, so that $\langle A\widetilde{A}\,\rangle_0 = \sum (A^{\underline{K}})^2 \geq 0$. Clearly we get equality iff all the squares are zero, that is $A = 0$. □

Lemma 4.3 *Any* $f - d$ *causal paravector* $U \in \mathcal{P} \subset \mathcal{C}\ell_p$ *can be written as* $U = B\widetilde{B}$, *with* $B \in \mathcal{C}\ell_p$

Proof. If U is $f-d$ timelike we can use the group $\$\Gamma_{1,p}$, which is isomorphic to $\Gamma^{+}_{1,p}$ and can be used to transform 1 to any $f-d$ timelike paravector U by $U = R(1)\widetilde{R}$. In case U is a null vector, we can write it as $U = a(1+e)$ where $e \in \mathcal{V}$ is a unit vector and $a > 0$. Then by observing that $(1+e)^2 = 2(1+e)$ we get that $U = V\widetilde{V}$, where $V = \frac{1}{\sqrt{2a}}U$. $\square$

This latter result reminds very much of the case with 2-spinors in [4], since there any null vector u^a was factorized as $u^a = \psi^A\overline{\psi}^{A'}$. The proof of the DP can now be given and is very similar to the one given with 2-spinors in [4],

Theorem 4.4 *For any multivector $A \in \bigotimes^r \mathcal{C}\ell_p$, $T\{A\}$ satisfies the dominant property.*

Proof. We need to prove that $\langle \widetilde{A}\overline{\underline{U}}A\,\rangle_0 \geq 0$ for all $f-d$ causal vectors U_i, with $\underline{U} = U_1 \otimes \cdots \otimes U_r$. By lemma 4.3 there are $B_i \in \mathcal{C}\ell_p$ such that if $\underline{B} = B_1 \otimes \cdots \otimes B_r$, then $\underline{B}\widetilde{\underline{B}} = B_1\widetilde{B}_1 \otimes \cdots \otimes B_r\widetilde{B}_r = \underline{U}$. Using lemmas 3.3 and 4.2 we get $\langle \widetilde{A}\overline{\underline{B}\widetilde{\underline{B}}}A\rangle_0 = \langle \widetilde{A}\widetilde{\overline{\underline{B}}}\overline{\underline{B}}A\rangle_0 = \langle \widetilde{\overline{\underline{B}}A}\,\overline{\underline{B}}A\rangle_0 \geq 0$. $\square$

So the fact that we have an Euclidean norm on $\bigotimes^r \mathcal{C}\ell_p$ makes the proof of the DP almost trivial. Other algebraic properties of these tensors are as simple to prove, we have

Proposition 4.5 *$T\{A\}$ vanishes if and only if $A = 0$.*

Proof. As remarked above $T\{A\}$ can be considered as belonging to $\bigotimes^r \mathcal{P}$. From lemma 4.2 we see that $T\{A\}_{00\ldots0} = \langle A\widetilde{A}\rangle_0 = 0$ iff $A = 0$. $\square$

Proposition 4.6 *If $T\{A\}(u_1,\ldots,u_r) = 0$ for some set of $f-d$ timelike vectors u_i, then $A = 0$.*

Proof. From the proof of lemma 4.3 we see that any $f-d$ timelike paravector U_i can be written $U_i = R\widetilde{R}$, with $R \in \$\Gamma_{1,p}$. But then $\langle \widetilde{A}\overline{\underline{U}}A\rangle_0 = \langle \widetilde{A}\overline{\underline{R}\widetilde{\underline{R}}}A\rangle_0 = \langle \widetilde{\overline{\underline{R}}A}\,\overline{\underline{R}}A\,\rangle_0$, where $\underline{R} \in \bigotimes^r \$\Gamma_{1,p}$. Using lemma 4.2 we get $\langle \widetilde{A}\overline{\underline{U}}A\,\rangle_0 = 0$ iff $\overline{\underline{R}}A = 0$. But $R\overline{R} \in \mathbb{R}_+$ so that $\underline{R}\overline{\underline{R}}A = 0$ iff $A = 0$. $\square$

Finally let us consider symmetries of our superenergy tensors. The following holds trivially

Proposition 4.7 *If $A \in \bigotimes \mathcal{C}\ell_p$ is symmetric or antisymmetric with respect to its components in some blocks, then $T\{A\}$ will be symmetric in the arguments in the corresponding blocks. Moreover if the paravector part of $A\widetilde{A}$ is symmetric in two blocks i, j, then $T\{A\}$ is symmetric in the i:th and the j:th arguments.*

It still remains to prove that the causal tensors in 4 dimensions, given by definition 3.5, also satisfy the above properties. But this is easy, it suffices to change lemma 4.2 by observing that we get $\langle \Psi\widetilde{\Psi} \rangle_0 = \sum A^{\underline{K}} A^{*\underline{K}} \geq 0$, where the asterisk denotes the complex conjugate. The other properties do not depend on the complex structure and hold as well.

Example 4.8 *(The metric revisited) Earlier we saw how the metric tensor could be generated as a superenergy in $\mathcal{C}\ell_3 f \otimes_{\mathbb{F}} \mathcal{C}\ell_3 f$. We will now consider the question for arbitrary dimension. To start with notice that the multivector*

$$G = 1\otimes 1 - e_1\otimes e_1 - e_2\otimes e_2 - e_3\otimes e_3 + e_{12}\otimes e_{12} + e_{13}\otimes e_{13} + e_{23}\otimes e_{23} - e_{123}\otimes e_{123}$$

in $\mathcal{C}\ell_3 \otimes \mathcal{C}\ell_3$ satisfies $G^2 = 8G$ and $\widetilde{G} = G$ due to that we have an even number of blocks. The paravector part of $G\widetilde{G}$ is

$$\langle G\widetilde{G} \rangle_{0\oplus 1} = 8\langle G \rangle_{0\oplus 1} = 8(1\otimes 1 - e_1\otimes e_1 - e_2\otimes e_2 - e_3\otimes e_3)$$

so that $\frac{1}{2\sqrt{2}}G$ generates the metric. Thus we do not need the complex structure to produce the metric. In fact we come to suspect that there might be various multivectors that generate the same superenergy, which is true since for example all simple multivectors generate a superenergy proportional to $T\{A\} = 1$. To continue with the metric, one can easily verify that for an arbitrary $\mathcal{C}\ell_p$ the multivector $G = \sum (-1)^{deg(I)} e_I \otimes e_I$, where the sum is over all multiindices, satisfies $G = \widetilde{G}$ and $G^2 = dim(\mathcal{C}\ell_p)G$. So $\frac{1}{\sqrt{dim(\mathcal{C}\ell_p)}}G$ generates the metric in all dimensions.

In the final section we investigate how the concept of principal null directions of causal tensors [15] can be treated within $\bigotimes^r \mathcal{C}\ell_p$.

4.2. PRINCIPAL NULL DIRECTIONS

In 4 dimensions there is an algebraic classification of spacetimes due to the following result on the Weyl tensor,

Theorem 4.9 *Every vacuum space-time admits at least one and at most four null directions* $l^a \neq 0$, $l^a l_a = 0$, *that satisfy*

$$l_{[a}C_{b]ef[c}l_{d]}l^e l^f = 0$$

This divides the set of possible spacetimes into different types, depending on how many different null directions there are. These null directions (called principal null directions PND) also satisfy

$$T_{abcd}l^a l^b l^c l^d = 0$$

where T_{abcd} is the Bel-Robinson tensor, the superenergy tensor of the Weyl tensor. This can be generalized to arbitrary superenergy tensors in arbitrary dimension by defining a principal null direction to $T\{A\}$ as a null vector l satisfying

$$T\{A\}_{ab\ldots c}l^a l^b \cdots l^c = 0$$

In [15] sufficient and necessary conditions for a nullvector l to be a PND of the superenergy tensor $T\{A\}$, $A \in \bigotimes^r \mathcal{C}\ell_{1,p}$, were obtained. It is nice that we can also do this for the superenergies of multivectors in $\bigotimes^r \mathcal{C}\ell_p$,

Theorem 4.10 *Let* $A \in \bigotimes^r \mathcal{C}\ell_p$ *then* $l \in \mathcal{P}$ *is a PND of* $T\{A\}$ *if and only if* $\underline{\bar{l}}A = 0$

Proof. Any null paravector l satisfies $l^2 = cl$ and $\widetilde{\bar{l}} = l$, where $c \neq 0$ is some real number. We then have $T\{A\}(l, l, ..., l) = \langle \widetilde{A}\underline{\bar{l}}A\rangle_0 = \frac{1}{c}\langle \widetilde{A}\overline{\underline{l}\,\underline{\tilde{\bar{l}}}}A\rangle_0 = \frac{1}{c}\langle \widetilde{\underline{\bar{l}}A}\underline{\bar{l}}A\rangle_0 = 0$ iff $\underline{\bar{l}}A = 0$ □

Thus we have a simple characterization of the PND's of $T\{A\}$. As corollaries of the above we also have the following

Corollary 4.11 *If* $(1 \otimes 1 \otimes \cdots \otimes \bar{l} \otimes \cdots \otimes 1)A = 0$ *then* l *is a PND of* $T\{A\}$. *Especially, if* $A \in \mathcal{C}\ell_p \otimes \cdots \otimes l\mathcal{C}\ell_p \otimes \cdots \otimes \mathcal{C}\ell_p$ *then* l *is a PND of* $T\{A\}$

Proof. We can write $\underline{\bar{l}} = (\bar{l} \otimes \cdots \otimes 1 \otimes \cdots \otimes \bar{l})(1 \otimes \cdots \otimes \bar{l} \otimes \cdots \otimes 1)$, so that $\underline{\bar{l}}A = 0$ if $(1 \otimes 1 \otimes \cdots \otimes \bar{l} \otimes \cdots \otimes 1)A = 0$. If $A \in \mathcal{C}\ell_p \otimes \cdots \otimes l\mathcal{C}\ell_p \otimes \cdots \otimes \mathcal{C}\ell_p$, we can write $A = (1 \otimes 1 \otimes \cdots \otimes l \otimes \cdots \otimes 1)B$ for some $B \in \bigotimes^s \mathcal{C}\ell_p$ and $\underline{\bar{l}}(1 \otimes 1 \otimes \cdots \otimes l \otimes \cdots \otimes 1) = 0$, since $\bar{l}l = 0$. □

which can be easily generalized

Corollary 4.12 *Any sum of multivectors* $A_i \in \mathcal{C}\ell_p \otimes \cdots \otimes l\mathcal{C}\ell_p \otimes \cdots \otimes \mathcal{C}\ell_p$ *(the* l *not necessarily in the same block for different* i*) has a superenergy tensor with* l *as a PND.*

Using this last corollary we can construct a large number of causal tensors having at least a given set of principal null directions. However we must point out that it might not be easy to classify a given A as being of the kind in the last corollary. It remains to try to find some kind of converse to these corollaries. Let us consider the case when we only have one block $\mathcal{C}\ell_p$.

Proposition 4.13 *Let* $A \in \mathcal{C}\ell_p$ *then* $\bar{l}A = 0 \Leftrightarrow A \in l\mathcal{C}\ell_p$*.*

Proof. The implication from right to left was proven above. We need to prove that $\bar{l}A = 0 \Rightarrow A \in l\mathcal{C}\ell_p$. To this end write $A = a_0 + a_1e_1 + ... + a_pe_p + a_{12}e_{12} + ... + a_{1p}e_{1p} + ... + a_{p-1p}e_{p-1p} + ... + a_{12...p}e_{12...p}$. We can then choose $l = 1 + e_1$ since in general $l = 1 + e$, $e^2 = 1$ and e can be used in constructing an ON-basis for $\mathbb{R}^p$. We can then set $e = e_1$ and write A in this basis. We get

$$\begin{aligned}
\bar{l}A \quad = \quad & a_0 + a_1e_1 + ... + a_pe_p + a_{12}e_{12} + ... + a_{1p}e_{1p} + ... + a_{p-1p}e_{p-1p} + ...+ \\
& a_{12...p}e_{12...p} - a_0e_1 - a_1 - a_2e_{12} - ... - a_pe_{1p} - a_{12}e_2 - ...- \\
& a_{1p}e_p - a_{23}e_{123} - ... - a_{p-1p}e_{1p-1p} - ... - a_{23...p}e_{12...p} - ...- \\
& a_{12...p}e_{2...p} \\
= \quad & (a_0 - a_1) + (a_1 - a_0)e_1 + (a_2 - a_{12})e_2 + ... + (a_p - a_{1p})e_p + ...+ \\
& (a_{23...p} - a_{12...p})e_{23...p} + (a_{12...p} - a_{23...p})e_{12...p}
\end{aligned}$$

since for all terms we get a term $a_{ij...k}e_{ij...k}$ and a term $-a_{1ij...k}e_{ij...k}$. We also get terms $-a_{ij...k}e_{1ij...k}$ and $a_{1ij...k}e_{1ij...k}$, thus $\bar{l}A$ vanishes iff $a_0 = a_1$ and $a_I = a_{1I}$ for all multiindices I not containing 1 so that A must be of the type

$$\begin{aligned}
A \;=\; & a_0 + a_0e_1 + a_2e_2 + a_2e_{12} + ... + a_pe_p + a_pe_{1p} + ... + a_{23...p}e_{23...p} + \\
+ \; & a_{23...p}a_{12...p} \\
= \; & a_0(1 + e_1) + a_2(1 + e_1)e_2 + ... + a_p(1 + e_1)e_p + ... + a_{23...p}(1 + e_1)e_{23...p}
\end{aligned}$$

which means that $A \in l\mathcal{C}\ell_p$ □

As a corollary we have

Corollary 4.14 *If* $A = A_1 \otimes \cdots \otimes A_r$ *with* $A_i \in \mathcal{C}\ell_p$ $\forall i$, *then* l *is a PND of* $T\{A\}$ *iff* $A_i \in l\mathcal{C}\ell_p$ *for some* i.

It would be desirable to extend these results to more general cases involving various blocks. However one will almost certainly not find such a strong converse as in the case of only one block due to that we, for example in the case of two blocks, can have combinations of the kind $A = B \otimes lC + lD \otimes E$.

5. Future Outlook

In coming work we will investigate the relation with earlier constructions of superenergy tensors. Especially we will prove that this construction contains all earlier ones, obtained with tensors or with Clifford algebra. In $\mathcal{C}\ell_3$ this construction also bears a natural relation to the superenergy of a Dirac spinor [15] and it is natural to treat a general multivector in $\bigotimes^r \mathcal{C}\ell_p$ as a spinor. It could also be interesting to look at the divergence and conservation of these new causal tensors in our construction, as well as to look further at the subject of principal null directions.

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