THE PARABOLIC ANALYTIC FUNCTIONS AND THE DERIVATIVE OF REAL FUNCTIONS

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(Received: May 11, 2004; Accepted: July 22, 2004)

Abstract. Among the bidimensional hypercomplex-number systems defined as $\{z = x + u\,y;\ u^2 = \alpha;\ x,\ y,\ \alpha \in \mathbf{R};\ u \notin \mathbf{R}\}$ the parabolic (dual) numbers are introduced with the rule $\alpha = 0$. As well as the functions of a complex variable, the analytic functions of a parabolic variable can be introduced as analytic continuation of the real functions of a real variable. These functions hold the property that the "imaginary" part is linked to the derivative of the "real" part.
In this paper we will show how this property allows one to demonstrate in an algebraic way some rules of the differential calculus for the real functions of a real variable.

1. Introduction

The three types of bidimensional hypercomplex variables can be defined by the algebraic ring

$$\{z = x + u\,y;\ u^2 = \alpha + u\beta;\ x, y, \alpha, \beta \in \mathbf{R};\ u \notin \mathbf{R}\}, \tag{1}$$

and they are two-dimensional examples of hypercomplex numbers [1, 2] or Clifford algebras [3, 4]. Their geometrical and physical relevance has been pointed out by Yaglom [2] and, after his book, in many papers[1]. For the purposes of the present paper, it will be sufficient to consider hypercomplex variables for which $\beta = 0$.

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[1] The main references are reported in [5, 6].

Advances in Applied Clifford algebras **14** No. 2, 185-190 (2004)

In this paper we report an interesting application of parabolic numbers [7] to differential calculus and, in particular, we show that many theorems about the derivative of the real functions of a real variable can be demonstrated in an algebraic way instead of the classical methods of mathematical analysis.

2. The Analytic Functions of a Parabolic Variable

It has been shown [1, 5] — depending on the sign of the real quantity α — that the bidimensional hypercomplex systems are rings isomorphic with one of the following three types (*canonical systems*):

1) for $\alpha < 0$ (Elliptic numbers); the canonical system is the system of complex numbers, with $u^2 = -1$.

2) for $\alpha = 0$ (Parabolic numbers).

3) for $\alpha > 0$ (Hyperbolic numbers); the canonical system is related to the bidimensional pseudo-Euclidean (space-time) geometry, with $u^2 = +1$.

Generally, one can introduce functions of an hypercomplex variable in any of these systems. In analogy with complex analysis, an analytic function can be obtained as the analytical continuation of the corresponding function of a real variable [4, 7]. As a matter of fact, given a function of the real variable x in its power series form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \,, \tag{2}$$

the corresponding analytic continuation $f(z)$ is found just by means of the algebraic substitution $x \to z = x + u\,y$, i.e., [4]

$$f(z) \equiv f(x + u\,y) = \sum_{n=0}^{\infty} a_n \left(x + u\,y\right)^n \,. \tag{3}$$

Thanks to the properties of the “imaginary” unit u, one can demonstrate that these series can be also written as $S_1 + uS_2$, where S_1 and S_2 are power series of real variables that are required to be absolutely convergent.

Still in analogy with the functions of a complex variable, one could alternatively define for the three systems above

$$f(z) = U(x,\, y) + u\,V(x,\, y) \tag{4}$$

as a function of the hypercomplex variable $z = x + u\,y$ if the functions U, V satisfy the following system of partial differential equations [8]:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}; \qquad \frac{\partial U}{\partial y} = \alpha\,\frac{\partial V}{\partial x}. \tag{5}$$

These conditions are obtained similarly to the Cauchy-Riemann equations for the functions of a complex variable [9] and can be regarded to as the Generalized Cauchy-Riemann conditions (GCR).

For elliptic and hyperbolic systems the functions U and V can be obtained by decomposition into idempotent basis [8, 10] and are given by:

$$\begin{aligned} U &= \tfrac{1}{2}[f(x+\sqrt{\alpha}\,y)+f(x-\sqrt{\alpha}\,y)]; \\ V &= \tfrac{1}{2\sqrt{\alpha}}[f(x+\sqrt{\alpha}\,y)-f(x-\sqrt{\alpha}\,y)]. \end{aligned} \tag{6}$$

For complex numbers these expressions are the well known: $U = \frac{1}{2}[f(z) + f(\bar{z})]$; $V = \frac{1}{2i}[f(z) - f(\bar{z})]$, $\bar{z}$ being the complex conjugate of z. The functions in eqs. (6) satisfy the GCR conditions in eqs. (5), as one can verify.

The function of a parabolic variable can be obtained from the expressions in eq. (6) in the limit $\alpha \to 0$. It ensues[2]:

$$U = f(x); \qquad V = y\, f'(x), \tag{7}$$

where, as usual, we have indicated with $f'(x)$ the derivative of the real function of a real variable $f(x)$. To stress the fact that we are dealing with functions of a parabolic variable, hereafter we set $u \to p$ (p represents the first letter of *parabolic*), so that we get

$$f(z) = f(x) + p\, y\, f'(x). \tag{8}$$

3. Derivation Rules for the Functions of a Real Variable

Thanks to eq. (8), we can write:

$$f'(x) = \frac{\mathrm{Im}\,[f(z)]}{\mathrm{Im}(z)},$$

$\mathrm{Im}(\cdot)$ representing the “imaginary” part of the parabolic quantity within parenthesis. The above expression allows one to demonstrate, by elementary algebra, many of the derivation rules for the functions of a real variable.

The demonstrations that follow apply to analytic parabolic functions and its “real” and “imaginary” parts. We will take as the definition of analytic parabolic function the series expansion in eq. (3), with $u^2 \equiv p^2 = 0$ [4, 7].

Let us indicate with $f(x)$, $g(x)$, and $h(x)$ three functions of a real variable and with $f(z)$, $g(z)$, and $h(z)$ their analytic parabolic continuations, as defined by eq. (8).

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[2] This result can also be obtained by the series expansion in eq. (3) as shown in [4, 7].

3.1. The derivative of the sum of two functions

Because

$$f(z) + g(z) = [f(x) + g(x)] + p\,y\,[f'(x) + g'(x)]\,,$$

it follows immediately that

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x) + g(x)] = f'(x) + g'(x). \tag{9}$$

The same property can be easily demonstrated for the difference of two functions as well.

3.2. The derivative of the product of two functions

By taking into account the property $p^2 = 0$ for parabolic numbers, we have

$$\begin{aligned} f(z) \cdot g(z) &= [f(x) + p\,y\,f'(x)][g(x) + p\,y\,g'(x)] \\ &= f(x)g(x) + p\,y\,[f'(x)g(x) + f(x)g'(x)]\,. \end{aligned}$$

This demonstrates that

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \tag{10}$$

3.3. The derivative of the quotient function

Let us consider the quotient of the analytic continuations of the functions $f(x)$ and $g(x)$:

$$\frac{f(z)}{g(z)} = \frac{f(x) + p\,y\,f'(x)}{g(x) + p\,y\,g'(x)}.$$

If we multiply the numerator and the denominator of the right-hand side by the conjugate parabolic function of $g(z)$, $(\overline{g(z)} = g(x) - p\,y g'(x))$, we obtain:

$$\frac{f(z)}{g(z)} = \frac{[f(x) + p\,y f'(x)]\,[g(x) - p\,y g'(x)]}{[g(x) + p\,y g'(x)][g(x) - p\,y g'(x)]} \equiv \frac{f(x)}{g(x)} + p\,y\,\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

The following relation is thus demonstrated:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}. \tag{11}$$

3.4. THE DERIVATIVE OF THE INVERSE FUNCTION

Let us indicate

$$f(z) = f(x) + p\,y f'(x) \equiv X + p\,Y.$$

Because of the analyticity of the inverse function, we know that

$$f^{-1}(z) = f^{-1}(x) + p\,y (f^{-1})'(x),$$

and the definition of the inverse function gives the identity

$$f^{-1}\left[f(z)\right] \equiv z.$$

Then, it follows

$$\begin{aligned} &f^{-1}\left[f(z)\right] = f^{-1}(X + p\,Y) = f^{-1}(X) + p\,Y\,(f^{-1})'(X) = \\ &= x + p\,y f'(x)\,(f^{-1})'[f(x)] \equiv x + p\,y. \end{aligned}$$

From the previous expression, it ensues

$$f'(x)\,(f^{-1})'[f(x)] = 1.$$

Since $x \equiv f^{-1}(X)$, the relation

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[f^{-1}(X)\right] = \left.\frac{1}{f'(x)}\right|_{x \equiv f^{-1}(X)} \tag{12}$$

is demonstrated.

3.5. THE DERIVATIVE OF THE FUNCTION OF A FUNCTION

Let us consider the composite function

$$h(z) = f[g(z)]$$

and set

$$g(z) = g(x) + p\,y\,g'(x) \equiv X + p\,Y.$$

By the analyticity of f, we have:

$$h(z) = f(X + p\,Y) = f(X) + p\,Y\,f'(X).$$

Since $X \equiv g(x)$ and $Y \equiv y\,g'(x)$, we obtain

$$h(z) = f[g(x)] + p\,y\,g'(x) f'[g(x)].$$

The following result is thus demonstrated

$$\frac{\mathrm{d}}{\mathrm{d}x} f[g(x)] = f'[g(x)]\,g'(x). \tag{13}$$

4. Conclusions

We have obtained by means of simple algebraic calculations some rules of the differential calculus concerning the derivation of functions of a real variable. This has been possible thanks to the property of the versor (or “imaginary” unity) p of parabolic numbers that acts as a first order infinitesimal quantity with the following difference: the square of the versor p is exactly zero, whereas in differential calculus the square of an infinitesimal quantity is considered smaller than the quantity itself.

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