TWO-DIMENSIONAL HYPERCOMPLEX NUMBERS AND RELATED TRIGONOMETRIES AND GEOMETRIES

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(Received: January 15, 2004; Accepted: March 07, 2004)

Abstract. All the commutative hypercomplex number systems can be associated with a geometry. In two dimensions, by analogy with complex numbers, a general system of hypercomplex numbers $\{z = x + u\,y;\ u^2 = \alpha + u\,\beta;\ x,\ y,\ \alpha,\ \beta \in \mathbf{R};\ u \notin \mathbf{R}\}$ can be introduced and can be associated with plane Euclidean and pseudo-Euclidean (space-time) geometries.

In this paper we show how these systems of hypercomplex numbers allow to generalise some well known theorems of the Euclidean geometry relative to the circle and to extend them to ellipses and to hyperbolas. We also demonstrate in an unusual algebraic way the Hero formula and Pytaghoras theorem, and show that these theorems hold for the generalised Euclidean and pseudo-Euclidean plane geometries.

Contents

Advances in Applied Clifford Algebras **14** No. 1, 47-68 (2004)

1. Introduction

Complex numbers are related to the Euclidean geometry: indeed their invariant (the module) is the same as the Pythagoric distance (Euclidean invariant) and their unimodular multiplicative group is the Euclidean rotation group. It is well known that these features allow the complex numbers to represent plane vectors.

In the same way hyperbolic numbers, an extension of complex numbers [1, 2] defined as

$$\{z = x + h\,y;\ h^2 = 1;\ x, y \in \mathbf{R}; h \notin \mathbf{R}\},$$

are related to space-time geometry [3]. Indeed their square module given by[1] $|z|^2 = z\tilde{z} \equiv x^2 - y^2$ is the Lorentz invariant of two-dimensional special relativity, and their unimodular multiplicative group is the special relativity Lorentz group [3, 4]. These relations have been used to extend [5] and to generalise [6] the two-dimensional special relativity.

Moreover the hyperbolic numbers have allowed a formalisation of the pseudo-Euclidean trigonometry with the same coherence as the Euclidean trigonometry and by this formalisation the pseudo-Euclidean triangles can be solved as the Euclidean ones [7]. In the same work it has also been shown that, in the pseudo-Euclidean plane, the equilateral hyperbolas play the same role and satisfy the same theorems of the circles in the Euclidean plane.

The hyperbolic numbers are the bidimensional (commutative) representative of the more general Clifford algebras [8, 9, 10, 11]. In recent years they have also been studied for their particular and important algebraic and geometric properties [11, 12].

In the present paper we extend the relation between the Euclidean and the pseudo-Euclidean planes using a general two-dimensional hypercomplex (or complex-type [13]) variable, i.e., the algebraic ring:

$$\{z = x + u\,y;\ u^2 = \alpha + u\,\beta;\ x, y, \alpha, \beta \in \mathbf{R};\ u \notin \mathbf{R}\}, \tag{1}$$

and we show that in the geometries generated by these numbers, the ellipses and general hyperbolas play the role that the circles and the equilateral hyperbolas respectively play in the Euclidean and in the pseudo-Euclidean planes [7].

Moreover we obtain, by an algebraic approach, the *generalised Hero formula* for calculating the triangle area and the *generalised Pythagoras theorem*, that hold for all the proposed geometries.

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[1] We call $\tilde{z} = x - h\,y$ the hyperbolic conjugate of z as for complex numbers.

Besides these specific results, we regard the method itself as important: in fact, in the geometries that derive from the bidimensional hypercomplex numbers we do not have the same intuitive vision that we have for the Euclidean geometry. Then the algebraic method for demonstrating the theorems is a fundamental aid.

The paper is organised in the following way: in section 2 we recall some basic concepts and the geometrical representation of the bidimensional hypercomplex number systems.

In section 3 we derive the circumscribed, inscribed and ex-inscribed conics to a triangle and we demonstrate the Hero formula and Pythagoras theorem.

In an appendix some numerical examples are reported.

2. The General Complex-Type Number System

2.1. BASIC CONCEPTS

The two-dimensional hypercomplex numbers (1) have been introduced by S. Lie [14] as a two-dimensional example of the more general class of the commutative hypercomplex number systems[2]. In more recent years the bidimensional hypercomplex systems in general form have been considered in detail [1, 2, 15, 13].

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[2] Hypercomplex numbers [14, 16] are defined by the expression: $x = \sum_{\alpha=0}^{N-1} e_\alpha x^\alpha$ where $x^\alpha \in \mathbf{R}$ are called *components* and $e_\alpha \notin \mathbf{R}$ units or *versors*, as in vector algebra. This expression defines an hypercomplex number if the versors multiplication rule is given by a linear combination of versors: $e_\alpha e_\beta = \sum_{\gamma=0}^{N-1} C_{\alpha\beta}^{\gamma} e_\gamma$ where $C_{\alpha\beta}^{\gamma} \in \mathbf{R}$ are called *structure constants* and define the characteristics of the system.

The versor product defines also the product of hypercomplex numbers. This product definition makes the difference between vector algebra and hypercomplex systems and allows to relate the hypercomplex numbers to finite and infinite [16, 6] Lie groups. In fact the vector product is not, in general, a vector, while the product of hypercomplex numbers is still an hypercomplex number; this is true for the division too, that for vectors does not exist while for hypercomplex numbers, in general, does exist. In the two-dimensional case we have $x = e_0 x^0 + e_1 x^1$ and if we put $e_0 \equiv 1$, $e_1 \equiv u$, $x^0 \equiv x$, $x^1 \equiv y$, $C_{00}^0 \equiv C_{01}^1 \equiv 1$, $C_{00}^1 \equiv C_{01}^0 \equiv 0$, $C_{11}^0 \equiv \alpha$, $C_{11}^1 \equiv \beta$, we obtain the number system represented in eq. (1). As it is well known, such units (both elliptic and hyperbolic) can be decomposed into idempotent basis u_i: u_+, u_-, where $u_+ = 1/2(1+u), u_- = 1/2(1-u)$, for which it results: $(u_i)^n = u_i$ and $u_+ u_- = 0$ [4, 9]. Such decomposition is very useful for the effective calculation of the functions of hypercomplex variable [15]

It has been shown [1, 2], depending on the sign of the real quantity

$$\Delta = \beta^2 + 4\alpha \equiv (\beta - 2\,u)^2, \tag{2}$$

that for any α, β these bidimensional systems are ring isomorphic with one of the following three types[3] (*canonical systems*):
1) for $\Delta < 0$ (Elliptic numbers); the canonical system is the system of complex numbers, with $u^2 = -1 \Rightarrow \Delta = -4$.
2) for $\Delta = 0$ (Parabolic numbers); and canonical system with $u^2 = 0$.
3) for $\Delta > 0$ (Hyperbolic numbers); the canonical system is related to the pseudo-Euclidean (space-time) geometry with $u^2 = +1 \Rightarrow \Delta = 4$.
We call:

- *Euclidean*, the geometry associated with complex numbers, and
- *pseudo-Euclidean*, the geometry associated with canonical hyperbolic numbers.

In the general case we call:

- *elliptic*, the geometries associated with general number systems with $\Delta < 0$, and
- *hyperbolic*, the geometries associated with general number systems with $\Delta > 0$.

In this paper we do not consider the *parabolic geometry* [4].

2.2. GEOMETRICAL REPRESENTATION

Let us now introduce a plane by analogy with the Gauss-Argand plane of the complex variable. In these *representative planes*, that are called with the names of the number systems, i.e., elliptic or hyperbolic, we associate the points $P \equiv (x,\ y)$ with the general bidimensional hypercomplex numbers $z = x + u\,y$. Their topology is the same of the Euclidean plane for elliptic numbers, and of the pseudo-Euclidean (space-time) plane for hyperbolic numbers [13, 4]. The hyperbolic plane is divided in four sectors [2] by the straight lines (*null lines*

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[3] The differences between the three systems derive from the possibility of executing the division. In fact for complex numbers (in general for elliptic numbers), the division is always possible unless $z\tilde{z} \equiv x^2 + y^2 = 0 \Rightarrow x = y = 0$. For parabolic and hyperbolic numbers the division is again not possible for numbers (z_0) with zero module (see paragraph 2.2), but from this condition does not follow $x = y = 0$ and for these numbers there exist some non-zero numbers whose product with z_0 is zero. Such numbers are called *divisors of zero*.

[4] This system has been widely studied by Yaglom [3], who has shown the link existing between the parabolic numbers and the Galileo group of classical kinematics.

[3]) $x+\frac{1}{2}(\beta\pm\sqrt{\Delta})y=0$. Following [3, pag. 179], a segment or line is said to be of the *first (second) kind* if it is parallel to a line through the origin located in the sectors containing the axis Ox (Oy)[5].

Let us also consider the hypercomplex conjugate of z, that we indicate by $\bar{z}$, given by [2][6]:

$$\bar{z}=x+(\beta-u)y. \tag{3}$$

Therefore we have $z+\bar{z}\equiv 2\,x+\beta\,y\in\mathbf{R}$.

The "square module" of z is introduced by the expression:

$$D\equiv z\bar{z}=x^2-\alpha\,y^2+\beta\,x\,y. \tag{4}$$

This real quantity is a definite quadratic form for $\Delta\leq 0$ and it is not definite (it is zero for x, y on the null-lines) for $\Delta>0$. It defines, in the representative planes, the square distance of a point from the origin of the coordinate axes. It must be taken in the quadratic form of eq. (4), that can be also negative [13, 7].

If we consider two points $P_i\equiv(x_i,\,y_i)$, $P_j\equiv(x_j,\,y_j)$, their square distance $D_{i.\,j}$ is given by an extension of eq. (4):

$$\begin{aligned}D_{i,\,j}&=(z_i-z_j)(\bar{z}_i-\bar{z}_j)\equiv z_i\bar{z}_i+z_j\bar{z}_j-z_i\bar{z}_j-z_j\bar{z}_i\equiv\\&\equiv(x_i-x_j)^2-\alpha(y_i-y_j)^2+\beta(x_i-x_j)(y_i-y_j).\end{aligned} \tag{5}$$

We call:

$$D_i\equiv z_i\bar{z}_i. \tag{6}$$

For hyperbolic numbers this square distance is positive for segment of the first kind, negative for segment of the second kind. In section 3 we shall exclude the points in a position for which $D_{i,\,j}=0$ (*null distance*)[7].

When the linear distance (a segment length or radial coordinate [5, 7]) has to de used, we follow [3, p. 180], [17, pag. 72] and put[8]

$$d_{i,\,j}=\sqrt{|D_{i,\,j}|}. \tag{7}$$

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[5] In physical applications of hyperbolic numbers a physical meaning can be given to the line kinds and they are called *timelike* and *spacelike* [3, 4, pag. 181] [18].

[6] z and $\tilde{z}$ are the solutions of the characteristic equation related to the number systems [16].

[7] The definition of distances (metric element) is equivalent to introduce the bilinear form of the *scalar product.* The scalar product and the properties of hypercomplex numbers allow to state suitable axioms [3, p. 245] and to give to the representative planes of elliptic and hyperbolic numbers the structure of a vector space.

[8] As a general rule we indicate the square segment lengths by capital letters, and by the same small letters the square root of their absolute value.

From the expression of eq. (5) it follows that the locus of points $P \equiv (x,\, y)$ that have the same distance from a fixed point $P_c \equiv (x_c,\, y_c)$ is given by the conic:

$$(x - x_c)^2 - \alpha(y - y_c)^2 + \beta(x - x_c)(y - y_c) = K. \tag{8}$$

Then in their representative planes these conics can be considered as an extension of the Euclidean circles[9]. Moreover it is shown in [3, 7] for the pseudo-Euclidean plane and later in this paper for the general geometries, that the conics (8) maintain many properties peculiar of the Euclidean circles. Then, following Yaglom [3, pag. 181], we shall call *"circles"* the conics represented by eq. (8). We call K the square *"semi-diameter"*, and $P_c \equiv (x_c,\, y_c)$ the centre of the conic, and in general we shall extend the Euclidean terminology by using the italic style included in quotation marks.

In the case $\Delta > 0$ (the conics are hyperbolas) K can be positive or negative; we call *hyperbolas of the first (second) kind if their tangent straight lines are of the first (second) kind* [18, p. 52] as it has been done for the straight line. For the first kind $K < 0$, for the second kind $K > 0$[10].

We define *"semi-diameter"* of the hyperbolas the quantity

$$k = \sqrt{|K|}. \tag{9}$$

The hyperbolas with common centre and square *"semi-diameter"* k^2 and $-k^2$ are said to be *conjugate.*

For fixed values of α, β we have a family of conics. These conics have the same coefficients of the quadratic terms; then, as it is known from analytical geometry, they have the same eccentricity, axes directions and, as far as the hyperbolas are concerned, the null-lines as asymptotes.

From another point of view we can associate an $(\alpha,\, \beta)$ algebra with a given conic, so that this one has the properties of a circle in the representative plane.

As for the complex and hyperbolic planes [7], the angular coefficient of a straight line, determined by the two points P_i, P_j, is given by:

$$m_{ij} = \frac{y_i - y_j}{x_i - x_j} \tag{10}$$

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[9] It is well known that in the Euclidean plane a circle is usually defined as *the locus of points which are at a fixed distance (r) from a given point P_c.*

[10] We could guess to attribute a kind to any curve and call of the first (second) kind a curve if its tangent straight lines are of the first (second) kind but a general curve (for example all the conics except the hyperbolas given by eq. 8) does have tangent straight lines of both the kinds. Then the hyperbolas of the type (eq. 8) have the peculiar property that the tangent straight lines to a given arm do have the same kind. This allows to attribute a kind, depending on the K sign, to hyperbola arms.

In the hyperbolic plane, from the previously given definition:
if $|2\,\alpha\,m-\beta|<\sqrt{\Delta}$ the straight line is of the first kind,
if $|2\,\alpha\,m-\beta|>\sqrt{\Delta}$ the straight line is of the second kind.
Now we are in position to generalise well known theorems of the Euclidean geometry.

3. Geometry and Trigonometry in Two-Dimensional Algebras

3.1. The *"circle"* for three points

In the Euclidean and in the pseudo-Euclidean planes the circles and the equilateral hyperbolas are determined by three points; also the general "*circle*" given by eq. (8) is determined by three points.
Let us consider the following problem:
given three points $P_i \equiv (x_i,\, y_i),\ i=1,\,2,\,3$ in the representative plane of a generalised two-dimensional hypercomplex number, let us find a *"circle"* of the type of eq. (8), that passes for these points.
These three points can be considered as the vertices of a triangle, then this problem is the generalisation of the well known Euclidean problem of finding a circumcircle of a triangle. In the representation on a Cartesian plane, the problem is solved if we know the centre $P_c \equiv (x_c,\, y_c)$ and the circle radius. Now the problem is the same and we can find x_c, y_c, K, by solving the system of three equations derived from eq. (8)[11]:

$$D_{n,c}=(z_n-z_c)(\bar{z}_n-\bar{z}_c)\equiv z_n\bar{z}_n+z_c\bar{z}_c-z_n\bar{z}_c-z_c\bar{z}_n=K \text{ for } n=1,\,2,\,3. \quad (11)$$

Subtracting the first equation ($n=1$) from the other two ($n=2,\,3$) we obtain two equations for the unknown z_c, $\bar{z}_c$. In matrix form:

$$\begin{pmatrix} \bar{z}_2-\bar{z}_1 & z_2-z_1 \\ \bar{z}_3-\bar{z}_1 & z_3-z_1 \end{pmatrix}\cdot\begin{pmatrix} z_c \\ \bar{z}_c \end{pmatrix}=\begin{pmatrix} \bar{z}_2 z_2-\bar{z}_1 z_1 \\ \bar{z}_3 z_3-\bar{z}_1 z_1 \end{pmatrix}, \quad (12)$$

with the solution

$$z_c=\frac{z_1\,z_2\,\bar{z}_2-z_1\,z_2\,\bar{z}_1+z_2\,z_3\,\bar{z}_3-z_3\,z_2\,\bar{z}_2+z_1\,z_3\,\bar{z}_1-z_1\,z_3\,\bar{z}_3}{z_1\,\bar{z}_2+z_3\,\bar{z}_1+z_2\,\bar{z}_3-z_2\,\bar{z}_1-z_1\,\bar{z}_3-z_3\,\bar{z}_2} \quad (13)$$

The solution for $\bar{z}_c$ is the hypercomplex conjugate of eq. (13). From one of eqs. (11), eq. (13) and the equation for $\bar{z}_c$, we find:

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[11] The following calculations have been obtained by means of the software *Mathematica* [19].

$$\begin{aligned} K = -\frac{(z_2-z_1)(\bar{z}_2-\bar{z}_1)(z_3-z_1)(\bar{z}_3-\bar{z}_1)(z_2-z_3)(\bar{z}_2-\bar{z}_3)}{(z_1\,\bar{z}_2+z_3\,\bar{z}_1+z_2\,\bar{z}_3-z_2\,\bar{z}_1-z_1\,\bar{z}_3-z_3\,\bar{z}_2)^2} \equiv \\ \equiv -\frac{D_{1,\,2}D_{1,\,3}D_{3,\,2}}{(z_1\,\bar{z}_2+z_3\,\bar{z}_1+z_2\,\bar{z}_3-z_2\,\bar{z}_1-z_1\,\bar{z}_3-z_3\,\bar{z}_2)^2}. \end{aligned} \tag{14}$$

Let us call Q^2 the denominator in eq. (14). It results:

$$\bar{Q} = -Q. \tag{15}$$

Let us calculate Q as a function of the points coordinates. We have:

$$\begin{aligned} &Q = z_1\,\bar{z}_2+z_3\,\bar{z}_1+z_2\,\bar{z}_3-z_2\,\bar{z}_1-z_1\,\bar{z}_3-z_3\,\bar{z}_2 \equiv \\ &\equiv (-\beta+2u)[x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1)]. \end{aligned} \tag{16}$$

From eq. (2), we can rewrite eq. (13) as:

$$z_c = \frac{(-\beta+2\,u)[D_1(z_3-z_2)+D_2(z_1-z_3)+D_3(z_2-z_1)]}{\Delta\ [x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1)]}. \tag{17}$$

The content of the square brackets can be recognised, except for the sign [7], as the double of the triangle area (S)[12]. Therefore, we can rewrite eqs. (14, 16) as:

$$K = -\frac{D_{1,\,2}D_{1,\,3}D_{3,\,2}}{4\,\Delta\cdot S^2}. \tag{18}$$

$$Q^2 = 4\,\Delta\cdot S^2 \tag{19}$$

Eq. (18) represents the generalised expression for the square *"semi-diameter"* of the circumcircle of an Euclidean triangle and for the circumscribed equilateral hyperbola in the pseudo-Euclidean geometry. In fact for the canonical systems

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[12] The triangle area is independent from the particular geometry. It is shown in [7] that it has the same value for the Euclidean as well as for the pseudo-Euclidean geometries. In the same way it can be shown that it is independent from α, β too.

eq. (18) becomes:

$$K = \begin{cases} \dfrac{D_{1,\,2}D_{1,\,3}D_{3,\,2}}{16\,S^2} \Rightarrow k = \dfrac{d_{1,\,2}d_{1,\,3}d_{3,\,2}}{4\,S} & \begin{array}{l}\text{Euclidean geometry}\\ (u^2=-1,\,\Delta=-4)\end{array} \\ -\dfrac{D_{1,\,2}D_{1,\,3}D_{3,\,2}}{16\,S^2} & \begin{array}{l}\text{pseudo-Euclidean}\\ \text{geometry}\\ (u^2=1,\ \Delta=4)\end{array} \end{cases} . \tag{20}$$

For hyperbolic numbers, K can be positive or negative, i.e., depending on the side kinds, the arms of circumscribed hyperbola are of the second or first kind.

3.2. THE HERO FORMULA AND PYTHAGORAS THEOREM

We know that in the Euclidean geometry the triangle area can be expressed as a function of the triangle sides lengths by means of Hero formula. Now, since the characteristic quantities are the square distances, we look for an expression of Q^2 as a function of the square side lengths, i.e.:

$$Q^2 = aD_{1,\,2}^2 + bD_{1,\,3}^2 + cD_{3,\,2}^2 + dD_{1,\,2}D_{1,\,3} + eD_{1,\,2}D_{3,\,2} + fD_{1,\,3}D_{3,\,2} \tag{21}$$

a, b, c, d, e, $f \in \mathbf{R}$. By substituting in eq. (21) the first expression of Q given in eq. (16), and the second expression of $D_{i,\,j}$ given in eq. (5), and comparing the coefficients of the left- and right-hand sides, we obtain a six equations system, whose solution is given by:

$$a = b = c = 1, \quad d = e = f = -2. \tag{22}$$

Then from eq. (19) we obtain

$$\begin{aligned} S^2 &= \tfrac{D_{1,\,2}^2+D_{1,\,3}^2+D_{3,\,2}^2-2(D_{1,\,2}D_{1,\,3}+D_{1,\,2}D_{3,\,2}+D_{1,\,3}D_{3,\,2})}{4\Delta} \equiv \\ &\equiv \tfrac{(D_{1,\,3}+D_{2,\,3}-D_{1,\,2})^2-4\,D_{1,\,3}\,D_{1,\,3}}{4\Delta} \end{aligned} \tag{23}$$

This equation gives the area squared of a triangle as a function of the square side lengths. For complex numbers (Euclidean geometry) this expression can be easily reduced to the product of four linear terms that represents the well known *Hero formula.* Then for the general number system the expression (23) will be called **generalised Hero formula**[13]. Let us discuss some consequences of eq. (23) and, for exemple, let us express eq. (23) in terms of the square distance $D_{3,\,2}$:

$$D_{3,\,2} = D_{1,\,3} + D_{1,\,2} \mp 2\sqrt{D_{1,\,2}D_{1,\,3} + \Delta \cdot S^2}\ . \tag{24}$$

If $D_{1,\,2}D_{1,\,3} + \Delta \cdot S^2 = 0$ eq. (24) becomes:

$$D_{3,\,2} = D_{1,\,3} + D_{1,\,2} \tag{25}$$

that, for the Euclidean and the pseudo-Euclidean [7] geometries, is the **Pythagoras theorem** for a right triangle. For a general algebra, if we have $D_{1,\,2}D_{1,\,3} + \Delta \cdot S^2 = 0$ we say that the sides $P_1\,P_2$ and $P_1\,P_3$ are *"orthogonal"* [14] and *the Pythagoras theorem holds for all the two-dimensional hypercomplex systems.* As already pointed out in [13] this theorem holds for all the systems in the form of eq. (25), i.e., with a sum in the right-hand side, but with the square distances that may be negative.

We conclude this section observing that *the reciprocal link between two-dimensional geometries and hypercomplex numbers has allowed to demonstrate in an unusual algebraic way the Hero formula and, as a consequence, the Pythagoras theorem. This demonstration holds for elliptic geometries, as well as for hyperbolic ones.*

3.3. PROPERTIES OF *"orthogonal"* LINES IN GENERAL ALGEBRAS

By using the Pythagoras theorem (eq. (25)), we can now derive the relation between the angular coefficients of the *"orthogonal"* straight lines determined by the points $P_1\,P_2$ and $P_1\,P_3$.

Using the points coordinates, eq. (25) becomes:

$$(x_3 - x_2)^2 - \alpha(y_3 - y_2)^2 + \beta(x_3 - x_2)(y_3 - y_2) = (x_3 - x_1)^2 - \alpha(y_3 - y_1)^2 + \\ + \beta(x_3 - x_1)(y_3 - y_1) + (x_1 - x_2)^2 - \alpha(y_1 - y_2)^2 + \beta(x_1 - x_2)(y_1 - y_2) \tag{26}$$

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[13] In the appendix A an alternative demonstration is given.

[14] We extend from the Euclidean to all the geometries, the word *"orthogonal"*. The definition of *"orthogonal"* line, here reported, is the same as that of differential geometry, i.e., two vectors are *"orthogonal"* if their scalar product (that depends on the metric) is null.

and the problem is put:
given the straight line $P_1 \leftrightarrow P_2$, find the coordinates of $P_3 \equiv (x_3,\, y_3)$ satisfying eq. (26).
Simplifying eq. (26) and substituting the angular coefficients given by eq. (10), we obtain the relation:

$$m_{12} = \frac{\beta\, m_{13} + 2}{2\alpha\, m_{13} - \beta}. \tag{27}$$

Eq. (27) can be written in a form that shows at once the reciprocity relation between m_{12} and m_{13}:

$$2\alpha\, m_{12}\, m_{13} - \beta(m_{12} + m_{13}) = 2 \qquad \text{or} \qquad (2\,\alpha\, m_{12} - \beta)(2\,\alpha\, m_{13} - \beta) = \Delta \tag{28}$$

From the second relation it follows that *if a straight line is of one kind* (sec. 2.2)*, its "orthogonal" straight line is of the other kind.*
For the canonical systems $\beta = 0$ and $\alpha = \mp 1$ we obtain the well known relations of the Euclidean and pseudo-Euclidean geometries:

$$m_{12} = \frac{1}{\alpha\, m_{13}} \equiv \mp \frac{1}{m_{13}}. \tag{29}$$

3.4. *"Incircles"* AND *"excircles"* OF A TRIANGLE

Let us consider three points and the three straight-lines between them; the problem proposed is the extension to geometries associated with a general bidimensional algebra, of the well known Euclidean problem of finding the incircle and excircles of a triangle, i.e., to construct a circle inside the triangle formed by the three lines (incircle) and the three circles, outside of the triangle, tangent to the straight lines (excircles). The solution of this problem shows which properties are preserved going from the Euclidean geometry to the general geometries and which properties must be considered peculiar of the Euclidean geometry. On the other hand we can not construct an hyperbola inside a triangle, but we shall see that, solving together the problem relative to *"incircles"* and *"excircles"*, we can find four *"circles"* with the properties of the four Euclidean circles (figures 1-4 in the appendix). Then we consider the problem of:
finding the *"circles"* with their centres equidistant from the straight lines.
This problem has a solution; moreover the *"circles"* that we shall find, have also other properties of the corresponding Euclidean circles. These other properties can be demonstrated by simple calculations, as it is shown in [7] for equilateral hyperbolas in the pseudo-Euclidean plane.

We note that for hyperbolic plane we have to consider, in general, both the conjugate arms of the hyperbolas. In fact the square distance from the straight line and the centre of the hyperbolas can be positive or negative depending on the sides (straight lines) kind, as we shall see in the next eqs. (30), (31).
For the problem solution we shall find the centre $P_c \equiv (x_c,\, y_c)$ and the *"semi-diameter"* $k \equiv \sqrt{|K|}$ of the "*incircle*" and "*excircles*".
This problem is reduced to the solution of two linear equations, as in the Euclidean geometry. Let us begin with the pseudo-Euclidean geometry.

Distance between a point and a straight-line in the pseudo-Euclidean plane

In the Euclidean geometry the distance $d_{c,\,\gamma_E}$ between a point $P_c \equiv (x_c,\, y_c)$ and a straight line γ_E : $\{\sin\phi\, x + \cos\phi\, y + q = 0\}$, is equal to the value which we obtain substituting the coordinates of P_c in the equation of γ_E. The elementary demonstration is obtained by using the property that this distance is the length of the segment $P_c\, P_N$, where P_N is the intersecting point between γ_E and the straight line from P_c orthogonal to γ_E. It has been shown in [7] that the same points P_c and P_N define the square distance $D_{c,\,\gamma_H}$ in the pseudo-Euclidean plane. Taking into account the plane topology the same expression of distance holds in the pseudo-Euclidean geometry.
In fact let be given the straight-line of the first kind $\gamma_H : \{\sinh\theta\, x + \cosh\theta\, y + q = 0\}$. Following the same procedure of the Euclidean analytical geometry, we obtain the square distance $D_{c,\,\gamma_H}$:

$$D_{c,\,\gamma_H} = -(\sinh\theta\, x_c + \cosh\theta\, y_c + q)^2. \tag{30}$$

If the straight line is a line of the second kind $\gamma_H : \{\cosh\theta\, x + \sinh\theta\, y + q = 0\}$ we obtain:

$$D_{c,\,\gamma_H} = (\cosh\theta\, x_c + \sinh\theta\, y_c + q)^2. \tag{31}$$

In both cases (eqs. 30 and 31) the distance

$$d_{c,\,\gamma_H} \equiv \sqrt{|D_{c,\,\gamma_H}|} \tag{32}$$

is a linear function of x_c, y_c.
In the Euclidean geometry we usually take the positive quantity:

$$d_{c,\,\gamma_E} = |\sin\phi\, x_c + \cos\phi\, y_c + q|, \tag{33}$$

as the distance between the point P_c and the straight line γ_E. In fact the quantity $\sin\phi\, x_c + \cos\phi\, y_c + q$ can be positive or negative depending on the position of the point in respect of the straight line. This last condition also holds for the pseudo-Euclidean plane.

The introduction of the extended hyperbolic functions [7, 5], allows to write equations of straight lines of both kinds by means of the expression γ_H : $\{\sinh_e \theta\, x + \cosh_e \theta\, y + q = 0\}$; then in the pseudo-Euclidean geometry as in the Euclidean geometry, we take the same expression of eq. (33) with the absolute value:

$$d_{c,\,\gamma_H} = |\sinh_e \theta\, x_c + \cosh_e \theta\, y_c + q|, \tag{34}$$

More precisely the absolute value in eq. (32) removes the sign in eqs. (30, 31) that depends on the straight line kind. The absolute value on the linear distance (34) removes the sign that depends on the different position of the point in respect to the straight line[15].

Let us return to the original problem and consider the three straight lines determined by the three points P_1, P_2, P_3 and let us denote γ_i : $\{\sinh_e \theta_i\, x + \cosh_e \theta_i\, y + q_i = 0\}$ the straight line between the points P_j, P_k, $i \neq j, k$. Introducing two quantities ϵ_1, ϵ_2 equal to ±1, that take into account the elimination of the absolute values in the expressions of the linear distances, we find the centres of the four equilateral hyperbolas by the equations:

$$d_{c,\,\gamma_1} = \epsilon_1\, d_{c,\,\gamma_2}, \qquad d_{c,\,\gamma_1} = \epsilon_2\, d_{c,\,\gamma_3} \tag{35}$$

From the centre coordinates we obtain the *"semi-diameters"* of hyperbolas by one of eq. 34.

The solution for general algebras

All the considerations we have seen so far for the Euclidean and for the pseudo-Euclidean geometries can be applied to general algebras.

Let us consider a straight line in general form γ_i : $\{y - m_i\, x - c_i = 0\}$ and a point $P_c \equiv (x_c,\, y_c)$ outside the straight-line. If we define, from eq. (5), the function of m_{ij}:

$$f(m_{ij}) \equiv \frac{D_{i,\,j}}{(x_i - x_j)^2} = 1 - \alpha(m_{ij})^2 + \beta m_{ij}, \tag{36}$$

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[15] For the problem solution this last sign can be preserved with a simple geometrical interpretation. In fact, for the Euclidean geometry, we note that the centres of excircles, in respect to the incircle, are in an opposite side of one straight line, then since we shall find the centres of the four circles by means of the same linear equations, we have to equate the distances but for the sign, and this is what we do when we equate the absolute values.

with $m_k = m_{ij}$ $(k \neq i,\, j)$, we obtain, as for the canonical systems, that the square distance from P_c and γ_i is proportional to the result of substituting the coordinates of P_c in the equation of γ_i.

$$D_{c,\gamma_i} = -\frac{\Delta \cdot (y_c - m_i\, x_c - c_i)^2}{4\, f(m_i)}, \tag{37}$$

and, also in this case, the distance is a linear function of x_c and y_c:

$$d_{c,\gamma_i} = \frac{1}{2}\sqrt{\left|\frac{\Delta}{f(m_i)}\right|}\,|(y_c - m_i\, x_c - c_i)|. \tag{38}$$

As for the Euclidean and pseudo-Euclidean planes, the sign of $y_c - m_i\, x_c - c_i$ is determined by the position of the point P_c with respect to the straight line γ_i.
In our specific problem we have the straight lines

$$\gamma_i\ : \{y - y_j = \frac{y_k - y_j}{x_k - x_j}(x - x_j)\ |\ \ i \neq j \neq k\} \tag{39}$$

between the points P_j and P_k[16].
Now we solve the system (35) with $d_{c,\,\gamma_i}$ given by eq. (38) and m_i, c_i given by eq. (39). By putting $\epsilon_3 = \epsilon_1\, \epsilon_2$, we obtain:

$$\begin{array}{rcl} x_c & = & \frac{\epsilon_1\, x_1\, d_{2,\,3} - \epsilon_2\, x_2\, d_{1,\,3} + \epsilon_3\, x_3\, d_{1,\,2}}{\epsilon_1\, d_{2,\,3} - \epsilon_2\, d_{1,\,3} + \epsilon_3\, d_{1,\,2}} \\ y_c & = & \frac{\epsilon_1\, y_1\, d_{2,\,3} - \epsilon_2\, y_2\, d_{1,\,3} + \epsilon_3\, y_3\, d_{1,\,2}}{\epsilon_1\, d_{2,\,3} - \epsilon_2\, d_{1,\,3} + \epsilon_3\, d_{1,\,2}} \\ K & = & -\frac{\Delta \cdot S^2}{(\epsilon_1\, d_{2,\,3} - \epsilon_2\, d_{1,\,3} + \epsilon_3\, d_{1,\,2})^2} \Rightarrow k = \\ & = & \sqrt{|K|} \equiv \frac{\sqrt{|\Delta|} \cdot S}{|\epsilon_1\, d_{2,\,3} - \epsilon_2\, d_{1,\,3} + \epsilon_3\, d_{1,\,2}|} \end{array} \tag{40}$$

We have obtained by means of the same equations the centres and the *"semi-diameters"* of the *"incircle"* and *"excircles"*. By analogy with the Euclidean geometry, we define inscribed the *"circle"* with the smallest *"semi-diameter"* $(\epsilon_1 = -\epsilon_2 = \epsilon_3 = -1)$. We note that the expressions for k as a function of the side lengths are the same of the Euclidean geometry.
In the appendix (figs. 1-4) we show two examples relative to the elliptic and hyperbolic planes.

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[16] The straight line *"orthogonal"* to γ_i has an angular coefficient given by eq. (27): $m_N = \frac{2\,(x_j - x_k) + \beta\,(y_j - y_k)}{-\beta\,(x_j - x_k) + 2\,\alpha\,(y_j - y_k)}$.

4. Conclusions

All the commutative hypercomplex number systems can be associated with a geometry [16]. Unfortunately, none of the geometries associated with commutative number systems of more than two units corresponds to the multidimensional Euclidean geometry or to four-dimensional space-time geometry used to describe the physical world. On the contrary the two-dimensional hypercomplex numbers can represent the Euclidean plane geometry and the space-time (Minkowsky) plane geometry.

In this paper we have shown that these two geometries can be studied simultaneously by using the general bidimensional hypercomplex number system. Moreover this study allows to construct geometries in which the role of circles in the Euclidean geometry is played by ellipses and hyperbolas.

This approach allows also to demonstrate, in an unusual algebraic way, the Hero formula and Pythagoras theorem, and to extend these theorems to the general Euclidean and pseudo-Euclidean plane geometries.

This last observation allows the following consideration: it is well known that all the theorems of the Euclidean geometry are direct consequences of the starting axioms, and are usually demonstrated by geometrical observations. Instead we work on the complex numbers by an algebraic-analytical method. The coincidence of the obtained results derives from the Klein association of the geometries to groups [3], and from the fact that the Euclid rotation group is the same of the unimodular multiplicative group of complex numbers. This association has been largely used in the XIX century also in the extensions of the Euclidean geometry to the differential and to the non-Euclidean geometries. It is then logical, although unusual, that the geometry theorems can be demonstrated in an analytical way.

All these well known considerations about the Euclidean geometry, can be extended to space-time geometry associated with hyperbolic numbers. Now, since we do not have for the general geometries the same intuitive vision that we have for the Euclidean geometry, it is a very useful opportunity that we can demonstrate theorems by an algebraic way.

As a final remark we note that the use of a general hypercomplex variable, i.e., without specification if it is elliptic or hyperbolic, allows a unification of all the discussed geometries, and a demonstration of the theorems by just one analytical development.

Acknowledgements

We wish to thank Prof. Dino Boccaletti and Prof. Stefano Marchiafava of University of Rome “La Sapienza” for their helpful suggestions and fruitful discussions.

Appendix

A. A Demonstration of Hero Formula

In the text we have discussed why and how the generalised Hero formula has been derived. Here we give a simpler demonstration. We start from the identity:

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \equiv [(z_1 - z_3) - (z_2 - z_3)][(\bar{z}_1 - \bar{z}_3) - (\bar{z}_2 - \bar{z}_3)]. \tag{41}$$

By expanding the right-hand side and by taking into account definitions (5, 16), we obtain:

$$\begin{aligned} & D_{1,\,2} = D_{3,\,2} + D_{3,\,1} - 2(\bar{z}_2 - \bar{z}_3)(z_1 - z_3) - Q \Rightarrow \\ & \Rightarrow 2(\bar{z}_2 - \bar{z}_3)(z_1 - z_3) = -D_{1,\,2} + D_{3,\,2} + D_{3,\,1} - Q. \end{aligned} \tag{42}$$

By multiplying eq. (42) by its hypercomplex conjugate and by taking into account eq. (15) we have:

$$4\, D_{3,\,2}\, D_{3,\,1} = (-D_{1,\,2} + D_{3,\,2} + D_{3,\,1})^2 - Q^2. \tag{43}$$

By substitution of eq. (19) into eq. (43) we obtain the Hero formula (23). We note that this demonstration is another example of how the proposed approach allows to demonstrate theorems just by using identities as it has been extensively shown in [7].

B. Numerical Examples

In figs. 1-4 two applications are shown, that represent the results obtained in section 3 about the *“circumcircle”*, *“incircle”* and *“excircles”* of a triangle[17]. The first application is pertinent to elliptic geometry, the second one to hyperbolic geometry.

The *“circumcircles”* have the coordinates of the centres given by eq. (17) and the square *“semi-diameters”* given by eq. (14). The centres and *“semi-diameters”* of the *“incircle”* and *“excircles”* are obtained by eqs. (40). In particular the three points $P_1 \equiv (3,\, 4)$, $P_2 \equiv (6,\, 8)$, $P_3 \equiv (7,\, 3)$, are given.

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[17] These figures have been obtained by using the software *Mathematica* [19].

In fig. 1 we report the five ellipses obtained with the values of the parameters: $\alpha = -3,\ \beta = 1,\ \Delta = -11 < 0$.

In fig. 2-4 we report the five hyperbolas obtained with the values of the parameters: $\alpha = 3,\ \beta = 1,\ \Delta = 13 > 0$.

Following the convention of paragraph 2.2 we have $m_{1\,2} = 4/3,\ m_{1\,3} = -1/4,\ m_{2\,3} = -5$ and $|2\alpha\, m_{1\,2} - \beta| \equiv 7 > \sqrt{\Delta} \equiv \sqrt{13},\quad |2\alpha\, m_{1\,3} - \beta| \equiv 5/2 < \sqrt{\Delta},\ \ |2\alpha\, m_{2\,3} - \beta| \equiv 31 > \sqrt{\Delta}$.

Then the sides $\overline{P_1P_2}$ and $\overline{P_2P_3}$ (the straight lines $\gamma_3,\ \gamma_1$) are of the second kind, the side $\overline{P_1P_3}$ (the straight line γ_2) is of the first kind. The hyperbola tangent to the straight lines $\gamma_3,\ \gamma_1$ is of the second kind: $(x-x_c)^2-(y-y_c)^2 = k^2$, the hyperbola tangent to the straight line γ_2 is of the first kind: $(x-x_c)^2-(y-y_c)^2 = -k^2$.

As far as the circumscribed hyperbola is concerned, we report both the conjugate arms, though the problem is solved by an hyperbola of a specific kind. In this particular example $D_{1,\,2},\ D_{1,\,3} < 0,\ D_{2,\,3} > 0$; then from eq. (18) we have $K < 0$, i.e., as it can be noted from fig. 2, the circumscribed hyperbola is of the first kind: $(y - y_c)^2 - (x - x_c)^2 = k^2$.

As far as the inscribed and ex-inscribed hyperbolas are concerned, we show in the text that we have obtained, by means of the same equations, the inscribed and ex-inscribed hyperbolas and, by analogy with the Euclidean geometry, we define inscribed the hyperbolas with the smallest *"semi-diameter"*. As it can be noted in figs. (2-4), they have also other properties of the corresponding Euclidean circles, i.e., the inscribed hyperbola is tangent to the triangle sides (and not to the external straight line) and its centre is inside the triangle.

As far as the ex-inscribed hyperbolas are concerned, their centres are outside the triangle and they are tangent to a triangle side and to two external straight lines.

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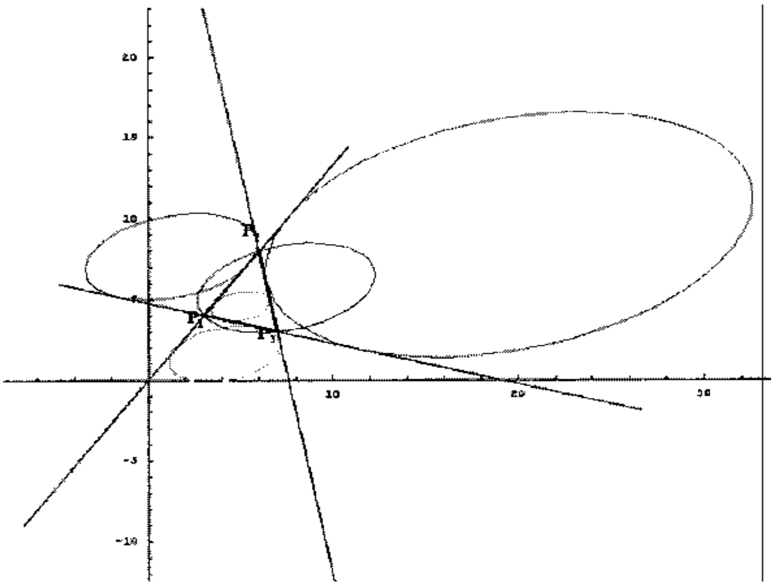

Fig. 1. Given the three points $P_1 \equiv (3,\,4)$, $P_2 \equiv (6,\,8)$, $P_3 \equiv (7,\,3)$, these points can be considered the vertices of a triangle.
In this figure we report the inscribed, circumscribed, and ex-inscribed ellipses associated with the elliptic algebra
$\{z = x + u\,y;\ u^2 = \alpha + \beta\,u;\ x,\,y \in \mathbf{R},\ \alpha = -3,\ \beta = 1,\ \Delta = -11\}$.
This example represents the generalisation of the incircle, circumcircle and excircles of an Euclidean triangle.

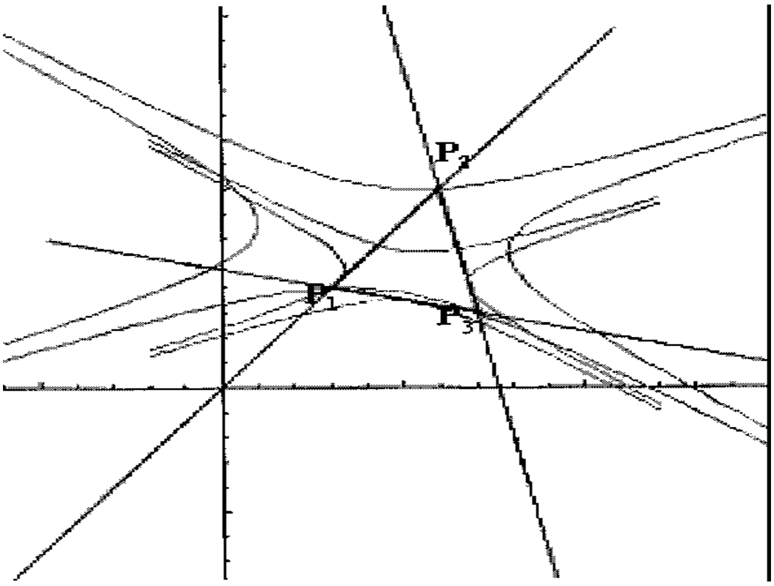

Fig. 2. Given the three points $P_1 \equiv (3,\, 4)$, $P_2 \equiv (6,\, 8)$, $P_3 \equiv (7,\, 3)$, these points can be considered the vertices of a triangle. In this figure we report the inscribed and circumscribed hyperbolas associated with the hyperbolic algebra $\{z = x + u\,y;\; u^2 = \alpha + \beta\,u;\; x,\, y \in \mathbf{R},\, \alpha = 3,\, \beta = 1,\, \Delta = 13\}$.

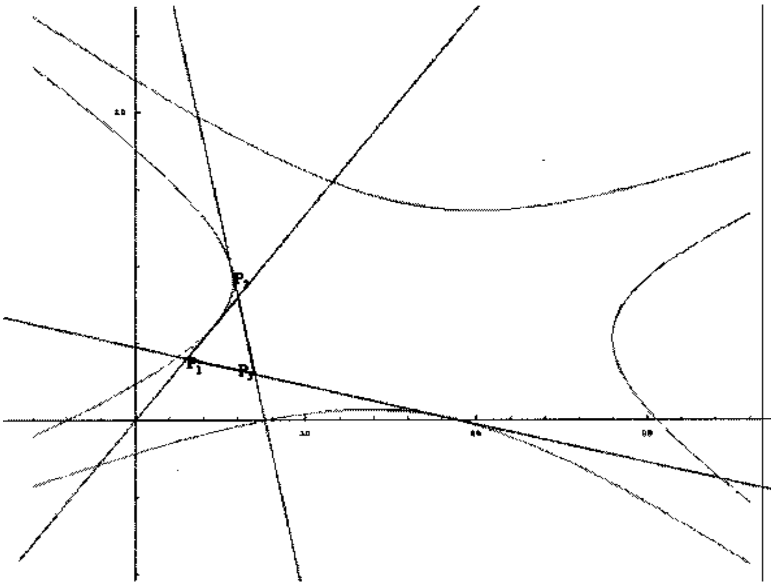

Fig. 3. The ex-inscribed hyperbolas of the triangle of fig. (2).

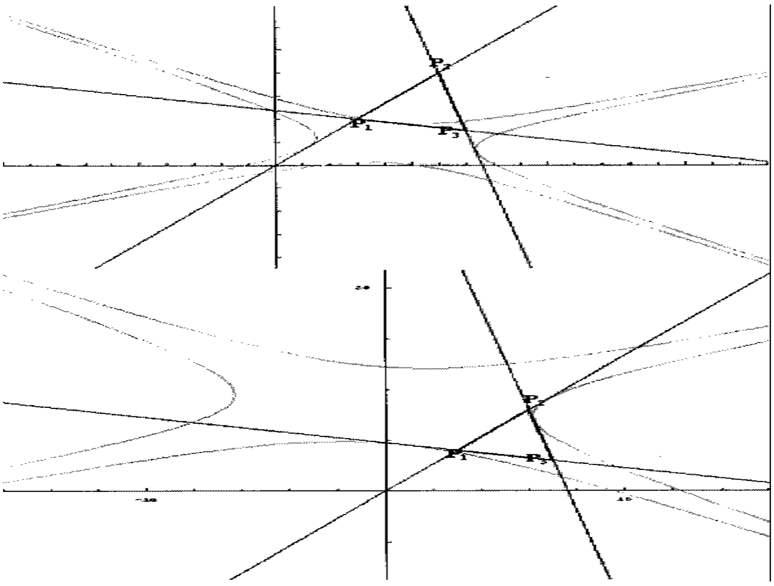

Fig. 4. The other ex-inscribed hyperbolas of the triangle of fig. (2).