PSEUDO UNITARY CONFORMAL GROUPS AND CLIFFORD ALGEBRAS FOR STANDARD PSEUDO HERMITIAN SPACES

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Abstract. This paper, self-contained, deals with pseudo-unitary spin geometry. First, we present pseudo-unitary conformal structures over a $2n$-dimensional complex manifold V and the corresponding projective quadrics $\tilde{H}_{p,q}$ for standard pseudo-hermitian spaces $H_{p,q}$. Then we develop a geometrical presentation of a compactification for pseudo-hermitian standard spaces in order to construct the pseudo-unitary conformal group of $H_{p,q}$. We study the topology of the projective quadrics $\tilde{H}_{p,q}$ and the "generators" of such projective quadrics. Then we define the space S of spinors canonically associated with the pseudo-hermitian scalar product of signature $(2^{n-1}, 2^{n-1})$. The spinorial group Spin $U(p,q)$ is imbedded into $SU(2^{n-1}, 2^{n-1})$. At last, we study the natural imbeddings of the projective quadrics $\tilde{H}_{p,q}$.

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<u>Foreword</u> The notion of spin structure on a manifold V has been introduced by A. Haefliger who specified an idea from Ehresmann, (*Sur l'extension du groupe structural d'un espace fibré*, C.R.A.S. Paris 243 (1956) p.558-560). J. Milnor, (*Spin structure on manifolds*, Enseignement Mathematique, Genève $2^{ème}$ série 9 (1963), p.198-203) and A. Lichnerowicz [18] have taken an interest in those structures. A.Crumeyrolle [36] has developed the study of associated vector bundles. Pertti Lounesto has studied conformal transformations in part 6 of his thesis, [40, b]. The problem of the investigation of real conformal spin structures on manifolds was made in [2, a, b, c] by the author of this paper.

Another article [2, d] deals, with real conformal symplectic spin geometry.

1. Pseudo-Unitary Conformal Structures

1 – Let V be an almost complex $2n$-dimensional paracompact manifold.
We know that any tangent space at V in a point x : T_x inherits a pseudo-hermitian structure of type (p, q), $p + q = 2n$ by the datum of f, pseudo-hermitian sesquilinear form of type (p, q). Such fields are differentially dependent on $x \in V$. We say that V is endowed with an almost pseudo-hermitian structure.
Any almost complex manifold inherits an almost pseudo-hermitian structure and an almost symplectic one.
Over an almost pseudo-hermitian manifold, the set of normalized orthogonal basis suitable for the almost pseudo-hermitian structure constitutes a principal bundle with structure group $U(p, q)$. (So, any almost complex manifold has its principal associated bundle of bases reducible to $U(p, q)$).
Conversely, as in [18, c] for the case of almost hermitian structures, one can show that if, over a $2n$-dimensional manifold, there exists a real 2-form of rank $2n$ F, there exists an almost pseudo-hermitian structure such that F be the fundamental 2-form and V inherits an almost pseudo-hermitian structure (and then, an almost complex structure).

2 – In any point $x \in V$, the tangent space T_x is equipped with a sesquilinear hermitian form f which determines the pseudo-hermitian scalar product of type (p, q).
T_x is so isomorphic to a standard space $H_{p,q}$ of type (p, q) with $p+q = 2n = n'$, defined in [5, b], ($H_{p,q} = (\mathbf{C}^{n'},\ f)$ f sesquilinear pseudo-hermitian form of type (p, q).
Let $\mathbf{C}^{n'}$, $n' = p + q$ be equipped with f.
We write $f(x, y) = R(x, y)+iI(x, y)$. We can verify that sesquilinearity implies that

$$\boxed{\begin{array}{l} R(ix, iy) = R(x, y)\ ; I(ix, iy) = I(x, y) \\ I(x, y) = R(x, iy) = -R(iy, x)\ ; R(x, y) = I(ix, y) = -I(x, iy) \end{array}}$$

and the hermitian character implies that

$$\boxed{R(x, y) = R(y, x) \text{ and } I(x, y) = -I(y, x)}$$

We know that there is identity between the datum of a complex vector space structure and that of a real vector space equipped with a linear operator J such that $J^2 = -\ Id$. $E = ({}_R E, J)$.

Thus $C^{n'} = (R^{2n'},\ J)$.
Let $x = \sum\limits_k (\xi^k + i\xi^{n'+k})e_k = \sum\limits_k x^k e_k \in C^{n'}$ identified with $\xi \in R^{2n'}$

and $y = \sum\limits_k^k (\eta^k + i\eta^{n'+k})\ e_k = \sum\limits_k y^k e_k \in C^{n'}$ identified with $\eta \in R^{2n'}$

Write $f(x,y) = \sum\limits_{j=1}^{p} x^j \bar{y}^j - \sum\limits_{j=p+q}^{p+q} x^j \bar{y}^j$ we find easily that,

$$f(x,y) = \sum_{j=1}^{p} (\xi^j \eta^j + \xi^{n'+j}\eta^{n'+j}) - \sum_{j=p+1}^{n'} \left(\xi^j \eta^j + \xi^{n'+j}\eta^{n'+j}\right) - $$
$$-i \left\{ \sum_{j=1}^{n'} \left(\xi^j \eta^{n'+j} - \eta^j \xi^{n'+j} \right) \right\}$$

Thus $\boxed{f(x,y) = (x,y) - i\lceil x|y\rceil}$

where (,) is a real bilinear symmetric form of type $(2p, 2q)$ and [|] is a classical symplectic real product. Id est that

$\boxed{U(p,q) = SO(2p,q) \cap Sp(2(p+q,R)}$ (cf.[5, b])

More precisely we have the following classical statement :

Theorem:

$U(p,q)$ is the set of elements $u \in SO(2p,2q)$ such that $uoJ = Jo$u.
$U(p,q)$ is the set of elements $u \in Sp(2(p+q),R)$ such that $uoJ = Jo$u.

3 – Let V be an almost pseudo-hermitian $2n$-dimensional manifold and let us denote T_x, the tangent space at V in $x \in V$ and $H(T_x)$ the real space of hermitian forms on T_x.

Definition:

A pseudo unitary conformal structure of type (p, q), $p \geq 0$, $q \geq 0$ $p + q = 2n = n'$ on V is the datum in any point x of a line C_x of $H(T_x)$ formed by the scalar multiples of a hermitian form of type (p, q) which satisfies the following lifting local axiom.
"There exists an open covering $(V_i)_{i \in I}$ of V and on any V_i an analytic section $y \in V_i \rightarrow h^i_y \in H(Ty)$ such that $h^i_y \in C_y$ for all $y \in V$".

Definition:

A conformal isometry from V onto V', both equipped with a pseudo-unitary structure of type (p, q) is an analytic diffeomorphism from V onto V' such that $\Phi(C_x) = C_{\Phi(x)}$ for any $x \in V$.

An almost pseudo-hermitian structure on V determines an associated conformal pseudo-hermitian structure of type (p, q). According to [5, b], p.230, the set of hermitian positive forms over T_x is a convex cone P of $H(T_x)$ and the set of strictly positive forms over T_x is a convex cone $\overset{o}{P}$ and $H(T_x) = P - P$.

2. Projective Quadric Associated with a Pseudo-Hermitian Standard Space $H_{p,q}$

Let $E = H_{p,q}$ be the standard pseudo-hermitian space C^{p+q} equipped with the classical pseudo-hermitian scalar product $f(x, y) = \sum_{i=1}^{p} x^i \bar{y}^i - \sum_{k=p+1}^{p+q} x^k \bar{y}^k$, the "unitary group" of which is called pseudo-unitary group of type (p, q) and denoted by $U(p, q)$.
The affine space associated with E inherits an almost pseudo-hermitian manifold structure by defining the scalar product in the vector space $E_x = x + E$, of vectors with origin x, by translation of that of E.

Let us introduce the hermitian quadratic form r associated with the pseudo-hermitian sesquilinear form f.

We know that r defined for any $x \in E$ by $\boxed{r(x) = f(x, x)}$ is such that $r(\lambda x) = |\lambda|^2 r(x)$, for all $\lambda \in C$.

Moreover $r(x) = f(x,x) = \sum_{i=1}^{p} \left|x^i\right|^2 - \sum_{k=p+1}^{p+q} \left|x^k\right|^2$
The function r takes real values.
We set $p+q=n'$.
Let us introduce the isotropic cone Q minus its origin, which constitutes a singular submanifold of $H_{p,q}=E$, defined by $\boxed{x \in Q \Leftrightarrow r(x)=0}$

Indeed, in any point $y \neq 0$ of a generator line $\mathbf{C}x$ of Q, the affine hyperplane T_y tangent at Q in y is identical with the hyperplane $T = y^{\perp}$ with equation :

$$x^1\bar{y}^1 + ..., +x^p\bar{y}^p - x^{p+1}\bar{y}^{p+1} ..... - x^{p+1}\bar{y}^{p+1} = 0$$

which is singular with radical $T^{\perp} \cap T = (y^{\perp})^{\perp} \cap T = \mathbf{C}y$.
An affine subspace S_x, with origin x, supplementary in T_x of the line $\mathbf{C}x$ is : $S_x = x + S$ translated of a supplementary S of $\mathbf{C}x$ in T. S is a regular space of type $(p-1, q-1)$. The natural map $u \in S \to u \bmod x \in (T/_{Cx})$ from S into the quotient space $T/_{Cx}$ is an isomorphism for hermitian quadratic forms and the vector subspace S_x of vectors with origin x is a regular subspace of T_x equipped with a hermitian form of type $(p-1, q-1)$ isomorphic to $T/_{Cx}$.
Let P be the classical projection from $E\backslash\{0\}$ into its associated projective space $P(E)$.

We assume that $x_1 \neq 0$

We can take $\left(\frac{x^2}{x^1}, ..., \frac{x^{p+q}}{x^1}\right)$ for coordinates at $\tilde{x} = P(x)$
Let $y = (y^1, \ldots, y^{p+q}) \in H_{p,q}$. We can express DP_x.

$$D \underset{x}{P}(y) = \left(\frac{y^2x^1 - y^1x^2}{\left(x^1\right)^2}, ..., \frac{y^{p+q}x^1 - y^1x^{p+q}}{\left(x^1\right)^2}\right)$$

We observe that the tangent vectors $\{y \text{ at } x\}$ and $\{(\lambda y) \text{ at } (\lambda x)\}$ have the same image, with $kerD \underset{x}{P} = \mathbf{C}x$
DP establishes natural linear isomorphisms : $D \underset{\lambda x}{P}$ from ($T_{\lambda x/Cx}$), $\lambda \neq 0$, onto $\mathrm{T}_{\tilde{x}}$ and $\mathrm{T}_{\tilde{x}}$ is equipped with a pseudo-hermitian form of signature $(p-1, q-1)$.

Definition:

The projective quadric $\tilde{Q} = \tilde{Q}(H_{p,q}) - \dim \tilde{Q} = p + q - 2$- is naturally equipped with a pseudo-unitary conformal structure of type $(p-1, q-1)$. By definition, such a quadric is called the projective quadric naturally associated with the hermitian space $H_{p,q}$. We agree to denote $\tilde{H}_{p,q}$ the projective quadric associated with $H_{p,q}$.

Remark:

Let us introduce ${}_RH_{p,q} = E_1$; the "realified" of $H_{p,q}$, - and the isotropic cone minus its origin $C^1_{2n'}$ of E_1. As $r(x) = 0$ for $x \in H_{p,q}$, is equivalent with $R(\xi, \xi) = 0$, we can identify the isotropic cone of $H_{p,q}$, with that of E_1, which has for equation :

$$\sum_{j=1}^{p} \left\{ (\xi^j) + \left(\xi^{n'+j}\right)^2 \right\} - \sum_{j=p+1}^{p+q} \left\{ (\xi^j)^2 + \left(\xi^{n'+j}\right)^2 \right\} = 0.$$

Introduce the natural projective space $P(E_1)$ associated with E_1 and the projective quadric $\tilde{Q}(E_1) = P(C^1_{2n'})$ in $P(E_1)$. $\tilde{Q}(E_1)$ is naturally equipped with a pseudo-remannian conformal structure of type $(2p-1, 2q-1)$. Such a quadric-real realization of $\tilde{Q}(H_{p,q})$ can be associated with $H_{p,q}$.

3. Conformal Compactification of Pseudo-Hermitian Standard Spaces $H_{p,q}, p + q = n$.

1- Let $H = H_{1,1}$ be the complex hyperbolic space equipped with an isotropic basis (ε, η) such that $f(\varepsilon, \eta) = 1$, [5], (f denotes the pseudo-hermitian form on H).

The direct orthogonal sum $F = H_{p,q} \oplus H = H_{p,q} \oplus H_{1,1}$ is a pseudo-hermitian standard space of type $(p+1, q+1)$. Let us introduce the isotropic cone $Q(F), \dim Q(F) = n+1$ and the projective quadric $\tilde{Q} = P(Q(F) - \{0\}) = M_1$ in the projective space $P(F)$ with dim $M_1 = n$.

Let us recall that $H_{p,q}$= C^{p+q} is identified with $R^{2(p+q)}$ according to the previous process, $R^{2(p+q)}$ equipped with the following basis : $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ orthogonal basis adapted to the complex structure determined by the R-linear map J such that $J^2 = -Id$. In the same way, we identify $H_{1,1}$ with R^4 of

type (2, 2) with the following basis $\{e_o, Je_o, e_{n+1}, Je_{n+1}\}$, orthogonal adapted basis such that $e_o^2 = 1 = (Je_o)^2$; $e_{n+1}^2 = -1 = (Je_{n+1})^2$.
The datum of $z = \alpha\ \varepsilon + x + \beta\eta \in F = H_{p,q} \oplus H$ with $\alpha\beta \in \mathbf{C}$ and $x \in H_{p,q}$ is equivalent with that of $Z = ae_o + be_{n+1} + cJe_o + dJe_{n+1} + x$ with $Z \in R^{2n+4}(2p+2, 2q+2), X \in R^{2n}(2p, 2q)$ and $a, b, c, d, \in \mathbf{R}$. Thus $z \in Q(F)$ is equivalent with $r(z) = 0$ id est $R(Z\ , Z) = 0$ id est :

$$R(X, X) + a^2 - b^2 + c^2 - d^2 = 0.$$

Moreover, $R(X, X) = f(x, x) = r(x) = Q_{2p,2q}(X)$ where $Q_{2p,2q}$ denotes the quadratic form naturally associated with the real symmetric bilinear form R. Thus $z \in Q(F)$ iff Z belongs to the isotropic cone of $R^{2n+4}(2p+2, 2q+2)$ id est iff $r(x) = Q(2p, 2q)(X) = b^2 - a^2 + d^2 - c^2$.
We can choose : $a = c = \frac{1}{2\sqrt{2}}(r(x) - 1)$ and $b = d = \frac{1}{2\sqrt{2}}(r(x) + 1)$. And introduce the map $\tilde{u} : X \mapsto\ \tilde{u}(X)$, where

$$\boxed{\begin{aligned}&\tilde{u}(X) = \\ &= \tfrac{r(x)}{2\sqrt{2}} \underbrace{(e_0 + J(e_0) + e_{n+1} + J(e_{n+1}))}_{\delta_0} + X + \tfrac{1}{2\sqrt{2}} \underbrace{(e_0 + J(e_0) - e_{n+1} - J(e_{n+1}))}_{\mu_0}\end{aligned}}$$

id est we introduce the following map p_1 from E into F

$$p_1\ : x \in E \to \boxed{p_1(x) = r(x)\delta_0' + x + \mu'_0}$$

where $\delta_0' = \frac{\delta_0}{2\sqrt{2}}$ and $\mu_0' = \frac{\mu_0}{2\sqrt{2}}$ such that $f(\delta_0', \delta_0') = 0 = f(\mu_0', \mu_0')$ and $f(\delta_0', \mu_0') = \frac{1}{2}$ $((\delta_0', \mu_0')$ constitutes an isotropic basis of $H_{1,1}$.

Definition:

> The projective quadric $M_1 = P(Q(F))$ image by P of $Q(F)$ in the corresponding projective space is called, by definition, the conformal compactified of $H_{p,q}$.

We are now going to justify such a definition.
Let $z = \alpha\delta_0' + x + \beta\mu_0'$ with $x \in H_{p,q}, \alpha, \beta \in \mathbf{C}$
$z \in Q(F)$ iff $f(z, z) = 0$ id est $\underline{\alpha\bar{\beta} + \bar{\alpha}\beta + 2r(x) = 0}$.

A vector $\mu = \alpha\delta_0^{'} + x + \beta\mu_0^{'}$ belongs to the tangent hyperplane at $Q(F)$ along the generator line Cz_o with $z_0 = \alpha_0^{'}\delta_0^{'} + x_0 + \beta_0\eta_0^{'}$ iff $\mu \in z_o^{\perp}$ id est iff α and β satisfy the relation : $\alpha\beta_0 + \bar{\alpha}_0\beta + 2f(x, x_0) = 0$. Let us introduce Vo the intersection of $Q(F)$ and of the affine hyperplane - (of F) - $\mu_0^{'} + (E \oplus C\delta_0^{'})$, y belongs to V_o iff $y = r(x)\delta_0^{'} + x + \mu_0^{'}$.
The map $p_1 \ : x \in E \to \mu_0^{'} + x + r(x)\delta_0^{'}$ is one to one from E onto V_o, and determines a bijective map between E and the generator lines of $Q(F)$ which do not belong to the hyperplane $T_\infty = E \oplus C\delta_0^{'}$, thus a one to one map from E onto $P(V)_o = V$ in the projective space $P(F)$.
V is an open set of the projective quadric M_1 and M_1 is topologically $\bar{V}$ (the closure of V) in $P(F)$. $W = M_1 - V$ is the image in $P(F)$ of the intersection Wo of $Q(F)$ with the hyperplane $T_\infty = E \oplus \mathbf{C}\delta_0^{'}$. T_∞ is singular with radical $T_{\infty,}^{\perp} \cap T_\infty = \mathbf{C}\delta_0^{'}$, and, so, tangent to the cone $Q(F)$ along tho isotropic line $\mathbf{C}\delta_0^{'}$.
W is a degenerate quadric of dimension $n-1$ in the projective hyperplane $\tilde{T}_\infty$: it is the "projective cone" formed by the projective lines with origine $\delta_0^{'} \in T_\infty$ resting against the regular projective quadric $\tilde{Q}\ (E)$ of dimension $n-2$ lying in the subspace $P(E)$ of $P(F)$. Indeed, $z = x + \lambda\ \delta_0^{'}$, with $x \in E$, belongs to $Q(F)$ iff $f(z, z) = 0 = f(x, x)$ and $W_0 = Q(E) + \mathbf{C}\delta_0^{'}$.
So, the conformal compactified M_1 of $E = H_{p,q}$ can be obtained by adjunction to E of a projective cone at infinity.
Let us determine D_{p_1} at $x \in E$
First, we note that for all $x,\ \ u \in Er\ (x + u) = r(x) + r(u) + 2R(x, u)$ with previous notations, as $f(x, u) + f(u, x) = f(x, u) + f(x, u) = 2R_e(f(x, u)) = 2R(x, u)$.
So, $p_1(x + u) - p_1(x) = u + R(x, u)\delta_0^{'} + r(u)\delta_0^{'}$ and then : $(D_{p_1})_x(u) = u + 2R(x, u)\delta_0^{'}$. $(D_{p_1})_x$ is a linear injective map and realizes a linear isomorphism from E_x onto $S_{p_1(x)}$ the tangent subspace at $p_1(x)$ to V_0.
$S_{p_1(x)}$ is a supplementary of the generator line $D_{p_1}(x)$ in the tangent hyperplane at $p_1(x)$ to the cone $Q(F)$.
Moreover, as $\delta_0^{'}$ is isotropic and orthogonal to E, $\ r((D_{p_1})_x u) = r(u)$. Thus, $(D_{p_1})_x$ realizes a "pseudo-hermitian isometry" from E onto S_x [conservation of the hermitian quadratic form]. p_1 is a pseudo-hermitian isometry from the almost pseudo-hermitian manifold E onto its image $V_0 \subset Q(F)$. If we consider Pop_1, where P is the classical projection onto the projective space, Pop_1 is a "pseudo-unitary conformal isometry" from E onto V.

4. Pseudo-Unitary Conformal Groups of Pseudo-Hermitian Standard Spaces $H_{p,q}$

Any element, u of the pseudo-unitary group $U(F) = U(H_{p+1,q+1}) = U(p+1, q+1)$ globally conserves the isotropic cone $Q(F)$ interchanging the generator lines and "isometrically" mapping the tangent hyperplane at y to $Q(F)$ onto the tangent hyperplane at $u(y)$ to $Q(F)$.
By going to the quotient space $P(F)$, $U(F)$ operates on the projective space by its image
$PU(F) = PU(p+1, q+1) = \frac{U(p+1,q+1)}{Z_{n+2}}$, where the center Z_{n+2} of $U(p+1, q+1)$ is constituted by the λI with $\lambda \in \mathbf{C}$ and $\lambda\bar{\lambda} = 1$ and will be denoted by $U(1)I$; $PU(F) = PU(p+1, q+1)$ globally conserves the projective quadric $M_1 = \tilde{Q}(F)$ and respects its pseudo-unitary conformal structure.

Definition:

We call by definition $PU(F) = \frac{U(p+1,q+1)}{U(1)I}$ the pseudo-unitary conformal group of $E = H_{p,q}$.

The pseudo-unitary group $U(p, q) = U(H_{p,q})$ can be naturally identified with the subgroup of elements u of $U(F)$ such that $u(\delta_0^{'}) = \delta_0^{'}$ and $u(\mu_0^{'}) = \mu_0^{'}$.
Thus $U(E) \cap U(1)I = \{I\}$.
If $u \in U(E)$:

$$p_1(u(x)) = \mu_0^{'} + u(x) + r(u(x))\delta_0^{'} = u(\mu_0^{'}) + u(x) + r(u(x))\delta_0^{'} = \\ = u(\mu_0^{'} + x + r(x)\delta_0^{'}) = u(p_1(x)) as\ r(u(x)) = r(x)$$

As $p_1 o u = u o p_1$, u globally conserves the image $p_1(E) = V_o \subset Q(F)$ and the restriction of u to V_o is an "isometry" of the almost pseudo-hermitian manifold V_o onto itself.
By getting to the projective space, $U(E) = U(p, q)$ can be identified with a subgroup of $PU(F)$ constituted by conformal automorphisms of M_1. $U(E)$ globally conserves the "projective cone at infinity" W.

1 – Translations of E

First, we remark that the group of isometries of the almost pseudo-hermitian manifold E, constituted by the translations $T(E)$ cannot appear as a subgroup of $U(F)$ as any operator different from zero of $T(E)$ changes the origin. On the other hand, while transferred by p_1 onto V_o the translations "become" a natural subgroup $T(V_o)$ of $U(F)$.

To any vector a of E corresponds an element t_a of $U(F)$ such that $t_a(p_1(x)) = p_1(x+a) = p_1 o t_a(x)$

Definition:

We set by definition
$t_a(\mu_0^{'}) = (t_a(p_1(o)) = p_1(a) = \mu_0^{'} + a + r(a)\delta_0^{'}, t_a(x) = x + 2R(x,a))\delta_0^{'}$
for all $x \in E, t_a(\delta_0^{'} = \delta_0^{'})$.

We can immediately verify that t_a respects the pseudo-hermitian scalar product of F ; thus $t_a \in U(F)$.
Moreover,

$$t_a(p_1(x)) = t_a(\mu_0^{'} + x + r(x)\delta_0^{'}) = \mu_0^{'} + a + r(a)\delta_0^{'} + x + 2R(x,a)\delta_0^{'} + r(x)\delta_0^{'}$$
$$= \mu_0^{'} + a + x + (r(x) + r(a) + 2R(x,a))\delta_0^{'} = \mu_0^{'} + a + x + r(x+a)\delta_0^{'}$$
$$= p_1(x+a) = p_1 o t_a(x).$$

t_a globally conserves V_o. Its trace on V_o is the image by p_1 of the translation by a in E and $t_{a+b} = t_a o t_b$.

2 – Dilatations of E and Pseudo-unitary group Sim $U(p,q)$

Let us consider, now a dilatation $k_1\ : x \mapsto \lambda x$ of $E = Hp,q$. We assume that λ is a strictly positive real.
Such a dilatation is a pseudo-unitary conformal transformation of E. We associate with k_1 the following operation h_λ of $U(F)$:

Definition:

Let $k_1\ : x \mapsto \lambda x$ of $E = H_{p,q}$, with λ a strictly positive real,
Set $h_\lambda\ (\mu_0^{'}) = \frac{1}{\lambda}\mu_0^{'}, h_\lambda(x) = x$ for all $x \in E$ and $h_\lambda(\delta_0^{'}) = \lambda\delta_0^{'}$

As $r(\lambda_x) = |\lambda|^2 r(x)$ and as λ is chosen to be a strictly positive real scalar.
$r(\lambda x) = \lambda^2 r(x)$
Thus $p_1(\lambda x) = \lambda h_\lambda(p_1(x))$ id est $p_1 o k_1(x) = \lambda h_\lambda(p_1(x))$
or equivalently : $h_\lambda o p_1 = \frac{1}{\lambda} p_1 o k_1$
V_o is not transformed into itself by h_λ but the image of $p_1(x)$ by h_λ belongs to the generator line $p_1(k_1(x))$ and h_λ determines a conformal isometry of M_1 which globally conserves V and W.
We know that the group of affine similarities $S(E_1)$ where $E_1 = {}_R H_{p,q} = R^{2p+2q}(2p,2q)$ is classically the product of its three subgroups : $T(E_1)$, $H(E_1)$

(dilatations $\xi \rightarrow \lambda\,\xi$ with $\lambda > 0$) and $0(2p, 2q)$ and that any element s of $S(E_1)$ can be uniquely written $s = h_\lambda o t_a o u$, with $\lambda > 0, u \in 0(2p, 2q), a \in E_1$ such that : for all $y \in E_1, s(y) = \lambda(a + u(y))$.
We introduce the following definition :

Definition:

We call affine pseudo-unitary similarity of $E = Hp,\ q$ any transformation of $E\ : s = k_\lambda\ o\ t_a o u$ where $u \in U(p, q),\ t_a\ \in T(E),\ k_\lambda$ dilatation of E with λ a strictly positive real. We define the affine pseudo-group of similarities as the group denoted by Sim $U(p, q)$ generated by such transformations of E.

• Let us now consider $s \in$ Sim $U(p, q)$. We associated with s the following element

$t_s \in U(F)\ : t_s = h_\lambda o t_a o u$, with previous notations. As $p_1 o u = u o p_1$ and $t_a o p_1 = p_1 o t_a,\ t_s o p_1(x) = h_\lambda o t_a o u[p_1(x)] = h_\lambda o t_a o p_1[u(x)] = h_\lambda o p_1[u(x) + a] = \frac{1}{\lambda} p_1[\lambda u(x) + \lambda a] = \frac{1}{\lambda} p_1(s(x))$, as $h_\lambda o p_1 = \frac{1}{\lambda} p_1 o k_1$.

On the hyperplane $T_\infty = E \oplus \mathbf{C}\ \delta_0^{'}, t_s(x + \beta\ \delta_0^{'}) = u(x) + \lambda[\beta + 2R(u(x), a)]\delta_0^{'}$ according to previous results. Thus, $t_s(T_\infty) \subset T_\infty$
Conversely, we can remark that the conditions for an element $v \in U(F), v(T_\infty) \subset T_\infty$ and $v\delta_0^{'}\ \in \mathbf{C}\ \delta_0^{'}$ are equivalent.

Indeed, $\mathbf{C}\ \delta_0^{'} = rad\ T_\infty$ and $T_\infty = (\delta_0^{'})^\perp$. The subgroup of $U(F)$ constituted by the elements v such that $v(T_\infty) \subset T_\infty$ is the isotropy group of the generator line $\mathbf{C}\ \delta_0^{'}$.
It contains $U(1) \cdot I$, with previous notations.
As $r(\lambda x) = |\lambda|^2 r(x)$. If $\mu\delta_0^{'} = \lambda\delta_0^{'}$ the respect of the pseudo-hermitian scalar product implies that $\lambda = |\lambda|\ > o, u(\mu_0^{'}) = \frac{1}{\lambda}\mu_0^{'} + a + \lambda r(a)\delta_0^{'}$ with $a \in E$ and if $x \in E, u(x) = w(x) + 2\lambda R(u(x), a)\delta_0^{'}$ with $w \in U(E)$.
Thus, $u = t_s$ with $s = k_\lambda o t_a o w$, with $\lambda\ > o$.
One can easily verify that $t_{s' o s} = t_{s'} o t_s$. The map $s \rightarrow t_s$ is so an isomorphism from Sim $U(p, q)$ onto the subgroup constituted by the elements of $U(F)$ which conserves the generator line $\mathbf{C}\ \delta_0^{'}$ of $Q(F)$. If we consider Pot_s then $s \mapsto Pot_s$ is an isomorphism from Sim $U(p, q)$ onto the isotropy group $S_{\tilde{\delta}_0^{'}}$ of the "point at infinity" $\tilde{\delta}_0^{'}$ in the group $PU(F)$.

3 – Moreover, the classical Witt-theorem can be applied to pseudo-unitary geometry [5a ; 6a, b]. Consequently, $PU(F)$ is transitive on M_1.

Theorem:

The pseudo-unitary conformal compactified M_1 of $E = H_{p,q}$ is identical with the homogeneous space $\frac{PU(F)}{\text{Sim } U(p,q)}$ quotient space of the projective unitary group of F : $PU(F)$ by Sim $U(p,q)$, the group of similarities of $H_{p,q}$.

In order to describe the action of $PU(F)$ on M_1, it is enough, for a pecular point m of M_1, to determine a transformation of $PU(F)$ which sends m onto $\delta_0^{'}$, the others being obtained by using the elements of the isotropy group $S_{\tilde{\delta}_0^{'}}$.
Let us introduce v_o the unitary symmetry of F relative to the unitary vector $\delta_0^{'} + \mu_0^{'} -$ (as $r(\delta_0^{'} + \mu_0^{'}) = r(\delta_0^{'}) + r(\mu_0^{'}) + 2R(\delta_0^{'}, \mu_0^{'}) = 1) -, v_0(\delta_0^{'}) = -\mu_0^{'}, v_0(\mu_0^{'}) = -\delta_0^{'}$ while $v_o(x) = x$ for all $x \in E$. We determine the action of v_o on a point $y = p_1(x) = \mu_0^{'} + x + r(x)\delta_0^{'}, v_0(p_1(x)) = -\delta_0^{'} + x - r(x)\mu_0^{'}$.

• If $r(x) \neq 0 - (r(x) \in \mathbb{R}))-$

$v_o(p_1(x)) = -r(x)\left[\mu_0^{'} - \frac{x}{r(x)} + \frac{\delta_0^{'}}{r(x)}\right]$. Set $x' = \frac{-x}{r(x)}$ such that $r(x') = \frac{1}{r(x)}$ and $p_1(x') = \mu_0^{'} + x' + r(x')\delta_0^{'} = \mu_0^{'} - \frac{x}{r(x)} + \frac{1}{r(x)}\delta_0^{'}$. We obtain : $v_o(p_1(x)) = -r(x)p_1(x')$.

• If $r(x) = 0, p_1(x)$ is sent by v_o into the hyperplane at infinity T_∞. The action of $\tilde{v}_o = P(v_o) \in PU(F)$ corresponds to the classical inversion with center at the origin and with power - 1 which sends "at infinity" all the points of the isotropic cone of $E = H_{p,q}$.
We notice that the inversion is not a transformation from E onto itself, on account of the existence of singular points, while its "realization" in M_1 is a conformal isometry of M_1 without any singular point.
We have just defined the inversion $I(0, -1)$ with center 0 and power -1 which appears while considering $x^{'} = -\frac{x}{r(x)}$ with $r(x') = \frac{1}{r(x)}$.
In the same way the inversion $I(0, 1)$ with center 0 and power 1 is $x \to x' = \frac{x}{r(x)}$.
Classically, for the real pseudo-orthogonal case, according to a theorem of Haantjes [39] which extends to pseudo-euclidean spaces of signature (r, s) with $r + s \geq 3$ the theorem of Liouville, the only real pseudo-euclidean orthogonal conformal transformations are the products of affine similarities and inversions.

As $H_{p,q}$ is identical with $(\mathbf{C}^{p+q}, f)$ and with $(\mathbf{R}^{\ 2(p+q)}\ , J)$ provided with a real bilinear symmetric form of type $(2p, 2q)$. According to the study of the corresponding pseudo-orthogonal conformal group $C_{2(p+q)}(2p, 2q)$ made in [2a, b, c], there cannot be other transformations than the previous ones in the pseudo-unitary conformal group of $H_{p,q}$.
Thus, we have obtained the following statement :

Proposition:

The conformal pseudo-unitary group of $H_{p,q} = E$ is the group constituted by products of conformal pseudo-unitary similarities and inversions of E.

5. Topology of the Projective Quadrics $\tilde{H}_{p,q}$

1 – Let $\{e_1, \dots e_{p+q}\}$ an orthogonal normalized basis which diagnonalizes the scalar classical pseudo-hermitian product of $H_{p,q}$. We denote by $\mathbf{H}$ the isotropic cone of $E = H_{p,q}$, with $n = p + q$ and the equation
$\sum\limits_{j=1}^{p} \left\{(\xi^j)^2 + (\xi^{n+j})^2\right\} - \sum\limits_{j=p+1}^{p+q} \left\{(\xi^j) + (\xi^{\ n+j})^2\right\} = 0,$
with previous notations. Let us introduce ${}_R H_{p,q} = E_1$; the "realified" of $H_{p,q}$, and the classical unit euclidean sphere of E_1 with radius $\sqrt{2}$ the equation of which is $\sum\limits_{j=1}^{p} \left\{(\xi^j)^2 + (\xi^{n+j})^2\right\} + \sum\limits_{j=p+1}^{p+q} \left\{(\xi^j)^2 + (\xi^{n+j})^2\right\} = 2$
Thus $x \in S \cap \mathbf{H}$ iff $\sum\limits_{j=1}^{p} \left\{(\xi^j)^2 + \left(\xi^{n+j}\right)^2\right\} = 1 = \sum\limits_{j=p+1}^{p+q} \left\{\left(\xi^j\right)^2 + \left(\xi^{n+j}\right)^2\right\}$
id est iff x belongs to the product of the unitary sphere $\sum_p$ of the standard hermitian space H_p by the unitary sphere $\sum_q$ of the standard hermitian space H_q. $\sum_p$ is classically isomorphic with S^{2p-1} and $\sum_q$ with S^{2q-1}. Let y be a point of $\mathbf{H}$ - $\{0\}$.
Necessarily we have :

$$\sum_{j=1}^{p} \left\{(\eta^j)^2 + (\eta^{n+j})^2\right\} - \sum_{j=p+1}^{p+q} \left\{(\eta^j)^2 + (\eta n^{n+j})^2\right\} = \rho > 0$$

The generator line $\mathbf{C}y$ is such that $\mathbf{C}y \cap\ (\sum_p \times \sum_q) = \left\{\frac{e^{i\varphi}}{\sqrt{\rho}}\, y\ ,\ \varphi \in R\right\}$. Conversely, any $(a, b) \in\ \sum_p \times \sum_q$ belongs to a generator line of H which it determines.

We have a natural map from $\sum_p \times \sum_q$ (or from $S^{2p-1} \times \ S^{2q-1}$) onto the projective quadric $\tilde{H}_{p,q}$ which enables us to identify $\tilde{H}_{p,q}$ with the quotient of the manifold $S^{2p-1} \times S^{2q-1}$ by the equivalence : $(a,b) \sim e^{i\varphi}(a,b)$ and thus realizes an $U(1)$- covering $(\sum_p \times \sum_q) = S^{2p-1} \times S^{2q-1}$ of kernel $U(1)$ of $\tilde{H}_{p,q}$. Consequently, $\tilde{H}_{p,q}$ is isomorphic with $\frac{S^{2p-1} \times S^{2q-1}}{S^1}$. We recall that S^{2p-1} is a bundle over P^{p-1} $(\mathbf{C})$ with typical fibre S^1. It is one of the Hopf classical fibrations [30, p.106, 107, 18 ; 31, p.75 ; 27 ; 28]. In fact, P^{p-1} $(\mathbf{C})$ is diffeomorphic with $\frac{U(p)}{U(p-1)xU(1)}$ and homeomorphic with $\frac{S^{2p-1}}{S^1}$. Thus $\tilde{H}_{p,q}$ is homeomorphic with $P^{p-1}(\mathbf{C}) \times S^{2q-1}$ and $S^{2p-1} \ \times P^{q-1}$ $(\mathbf{C})$. As p > 1, $q > 1$, we find again, as P^{p-1} $(\mathbf{C})$ is then simply connected [31, p.83] and as S^{2q-1} is simply connected that $\tilde{H}_{p,q}$ is simply connected, for $p > 1, \ q > 1$, [29, p. 124].

2 – <u>"Generators" of the projective quadrics $\tilde{H}_{p,q}$</u>

As for the pseudo-euclidean case, the maximal totally isotropic subspaces of $H_{p,q}$ contained in the cone $\mathbf{H}$ have complex dimension equal to inf (p,q). Their images in the projective space are the projective subspaces included in the projective quadric $\tilde{H}_{p,q} = P(\mathbf{H} - \{0\})$ which we agree to call "generators" of $\tilde{H}_{p,q}$ of complex dimension inf $(p,q) - 1$. Let us assume $p \geq q$.
Let E_- be the hermitian subspace of E with basis $\{e_1, \ldots, e_p\}$ and E_- be the anti-hermitian subspace of E with basis $\{e_{p+1}, \ldots, e_{p+q}\} E = E_+ \oplus E_-$. Any maximal totally isotropic subspace V of E canonically determines an antiisometry from E_- into E_+: for all, $t, t' \in E_-$:

$$f(\varphi_v(t), \varphi_v(t')) = -f(t,t').$$

V and E_- are both supplementaries of $E_+ \ : E = E_+ \oplus E_-, \ E = E_+ \oplus V$. If p_+ and p_- denote the restrictions to V of the projections on E_+ and E_- of the first decomposition, p_- is a linear isomorphism from V onto V_-. We take $\varphi_v = p_+ p_-^{-1}$. If $t \in \ E_- \ : p_-^{-1}(t) = t + p_+ p_-^{-1}(t) = t + \varphi_v(t) \in V$. For all, $t, t' \in E_-, f(t + \varphi_v(t), t' + \varphi_v(t') = 0.$ and φ_v is anti-isometry. We can associate with V the orthogonal system $U = \Phi(V) = \{u_1 = \varphi_v(e_{p+1}), \ldots, u_p = \varphi_v(e_{p+q})\}$of q. vectors of E_+.
Conversely with any orthogonal system U of q vectors $\{u_1, u_2, \ldots, u_q\}$ of E_+ we associate $V = \Psi(U)$ generated by the vectors $v = u_1 + e_{p+1}, v_2 = u_2 + e_{p+2}, \ldots \ , v_q = u_q + e_{p+q}$. The vectors $v_1, \ \ldots, \ v_q$ are linearly independent, isotropic, mutually orthogonal. V is, then, a maximal totally isotropic subspace.
Φ and Ψ are inverse mappings which determine a natural one to one mapping between the set of maximal totally isotropic subspaces of E - or, equivalently,

the set of "generators" of the projective quadric $\tilde{H}_{p,q}$ - and the Stiefel manifold $V_{p,q}$ of systems of q orthogonal vectors of the hermitian space H_p.
If $p > q$, such a manifold is identical with to quotient $\frac{U(p)}{U(p-q)} = \frac{SU(p)}{SU(p-q)}$ [27, p.89] is connected and simply connected.
If $p = q, \Phi$ establishes a one to one mapping from the set of "generators" of $\tilde{H}_{p,q}$ onto the set $V_{p,q}$ of orthogonal basis of H_p, which is identical with the unitary connected group $U(p)$, not simply connected with fundamental group classicaly isomorphic with Z.

6. Clifford Algebras and Clifford Groups of Standard Pseudo-Hermitian Spaces $H_{p,q}$

1 – We recall that $U(p,q)$ is the set of element $u \in S0(2p,2q)$ such that $uoJ = Jou$. (J : transfer-operator of the complex structure).
Let us introduce $C_{2p,2q}$ the real Clifford algebra of $E(2p,2q)$ the real pseudo-euclidean standard space equipped with a quadratic form of signature $(2p,2q)$.
$C_{2p,2q} = C^+_{2p,2q} \oplus C^-_{2p,2q}$, ($C^+_{2p,2q}$ even Clifford algebra and $C^-_{2p,2q}$= subspace of odd elements). $C^-_{2p,2q}$ can be seen as a $C^+_{2p,2q}$ module. We recall that $H_{p,q} \approx (E(2p,2q),J)$.

Theorem 1

There exists a linear mapping $\tilde{J}$ from $C_{2p,2q}$ into $C_{2p,2q}$ such that :
a) $C^+_{2p,2q}$ and $C^-_{2p,2q}$ are conserved by the action of $\tilde{J}$,
b) $\tilde{J}^2(c) = c, \forall c \in C^+_{2p,2q}$ and $\tilde{J}^2(c) = -c, \forall c \in C^-_{2p,2q}$,
c) $\tilde{J}(c_1 c_2) = \tilde{J}(c_1)\ \tilde{J}(c_2)$, for all $c_1,\ c_2\ \in C_{2p,2q}$.

We consider $\otimes\ E_{2p,2q}$ the tensor algebra of $E_{2p,2q}$ and we define the linear map J_1 from $\otimes E_{2p,2q}$ into $\otimes E_{2p,2q}$ by

•$J_1(x_1 \otimes \ldots \otimes x_k) = J(x_1)\ \otimes \ldots \otimes J(x_k)$

•$J_1(\lambda) = \lambda$ for all $\lambda \in R$.
J_1 is well defined. Let $N(Q_{2p,2q})$ be the two-sided ideal generated by the elements $x \otimes x - Q_{2p,2q}(x).1$, where $Q_{2p,2q}$ is the quadratic standard form of signature $(2p,2q)$ defined on $E_{2pq} ={}_R H_{p,q}$.

$$J_1\{x \otimes x - Q_{2p,2q}(x).1\} = J(x) \otimes J(x) - Q_{2p,2q}(x).1 = J(x) \otimes J(x) - Q_{2p,2q}(J(x)).1$$

as $Q_{2p,2q}(x) = Q_{2p,2q}(J(x))$. (We recall that J is orthogonal for $Q_{2p,2q}$).

J_1 conserves $N(Q_{2p,2q})$. So ${}_1J_1$ induces $\tilde{J}$, linear map from $C_{2p,2q}$ into itself which has the required properties. We can remark that $C^-_{2p,2q}$ is a **C**-space by setting for $c \in C^-_{2p,2q}, \lambda = \alpha + i\beta; \alpha, \beta \in \mathbf{R} : c(\alpha + i\beta) = c\alpha + \tilde{J}(c)\beta$.
($C^-_{2p,2q}$ is equipped with a transfer operator $\tilde{J}$ such that $\tilde{J}^2 = -\, Id$ on $C^-_{2p,2q}$).
We know that to any quadratic automorphism u of $E_{2p,2q}$, there canonically corresponds an automorphism Φ_u of $C_{2p,2q}$. If $u \in SO(2_p, 2_q), \Phi_u$ is an inner automorphism of $C_{2p,2q}$ and for all $x \in E_{2p,2q}, u(x) = \Phi_u(x) = b_u x b_u^{-1}$ where b_u is the product of an even number of regular vectors of $E_{2p,2q}$ and $b_u \in G^+_{2p,2q}-$ (the even Clifford group of $C_{2p,2q}$) -. More precisely, $u = \varphi(b_u)$ where $b_u \in G^+(E_{2p,2q})$ and φ denotes the natural homomorphism from $G(2_p, 2_q)$ onto $O(2_p, 2_q)$ associated with the exact sequence - (we recall that the Clifford group $G(2_p, 2_q)$ is the group constituted by invertible elements of the Clifford Algebra such that : for any g in
$G(2_p, 2_q)$, for any x in $E_{2p,2q}$, $\varphi(g)x = gxg^{-1} \in E_{2p,2q}$) - :

1. $\to R^* \to G(2_p,\ 2_q) \xrightarrow{\varphi} 0(2_p,\ 2_q)\ \to 1.$

   Moreover, we notice the following exact sequence :

2. $\to R^* \to G^+(2_p,\ 2_q) \xrightarrow{\varphi} S0(2_p, 2_q)\ \to 1.$

<u>Theorem 2</u>

U belongs to $S0(2_p, 2_q)$ and $Ju = uJ$ if and only if u induces an inner automorphism Φ_u of $C_{2p,2q}$ such that, for all $x \in E_{2p,2q}$ there exists $b_u \in G^+_{2p,2q}$ such that $\Phi_u(x) = b_u x b_u^{-1} = u(x)$ and $\tilde{J}(b_u) = b_u$.

• If $u \in S0(2_p, 2_q)$ and $Ju = uJ$, then, there exists $b_u \in G^+_{2p,2q}$, such that $\varphi_{b_u}(x) = b_u x b_u^{-1} = u(x), b_u = x_1 \ldots x_{2h}$, modulo a scalar in R^* where the x_i belong to $E_{2p,2q}$ and by definition of J which is a similarity of $(E_{2p,2q})$ of ratio $\rho = 1$ and of $\tilde{J}, \tilde{J}(b_u) = b_u$ as $\rho = 1$.

• Conversely, if u induces an inner automorphism of $C_{2p,2q}$ such that $\Phi_u(x) = b_u x b_u^{-1} = u(x)$ with $b_u \in G^+_{2p,2q}$, then necessarily $u \in S0(2_p, 2_q)$. As $\tilde{J}(b_u) = b_u$, we have $\tilde{J}(b_u^{-1}) = b_u^{-1}$. Then $u[\tilde{J}(x)] = b_u \tilde{J}(x) b_u^{-1} = \tilde{J}(b_u x b_u^{-1}) = \tilde{J}(u(x))$ and, so, $uJ = Ju$.
We notice that b_u is determined up to a factor in R^*.

<u>Definition 1</u>

We agree to call the Clifford algebra associated with $H_{p,q}$ the real algebra denoted by $Cl^{p,q} = \{g \in C^+_{2p,2q} : \tilde{J}(g) = g\} = \{z + \tilde{J}(z), z \in C^+_{2p,2q}\}$

If we choose the first definition, we notice that for all $g \in C^+_{2p,2q} z = g + \tilde{J}(z) \in Cl^{p,q}$ as $\tilde{J}(z) = z$. Then, we remark that if $z \in C^+_{2p,2q}$ and $\tilde{J}(z) = z$, $\frac{z}{2} + \tilde{J}(\frac{z}{2}) = z$ and $\frac{z}{2} \in C^+_{2p,2q}$, whence the result follows.
$Cl^{p,q}$ is defined as a subalgebra of $C^+_{2p,2q}$. $\tilde{J}$ is an involutive automorphism of $Cl^{p,q}$.

Definition 2

Let φ_J be the map from $H_{p,q}$ into $Cl_{p,q}$ defined by $\varphi_J(x) = x \cdot \tilde{J}(x) = x \cdot J(x)$.

• φ_J defines a map from $H_{p,q}$ into $Cl^{p,q}$ as $\tilde{J}(\varphi_J(x)) = \tilde{J}(x) \cdot \tilde{J}^2(x) = J(x) \cdot (-x) = x\tilde{J}(x)$ as $\tilde{J}^2(x) = -x$ and on the other hand, $x\tilde{J}(x) + \tilde{J}(x) \cdot x = 2R(x, J(x) = 0$, as $R(x, \tilde{J}(x) = -I(\tilde{J}(x),\ \tilde{J}(x) = 0$ where R denotes the bilinear real symmetric form associated with $Q_{p,q}$ and I the skew-symmetric form defining the symplectic product.

• φ_J is R-quadratic, which means that : $\varphi_J(\lambda x) = \lambda^2 \varphi_J(x)$ for all $x \in H_{p,q}$ and for all $\lambda\ \in R$, and $\frac{1}{2}\{\varphi_J(x+y) - \varphi_J(x) - \varphi_J(y)\} = \frac{1}{2}\{x \cdot J(x) + y \cdot J(x)\} = \varphi(x, y)$ where φ is an R-bilinear symmetric form from $H_{pq} x H_{p,q}$ into $Cl^{p,q}$. (The verification is easy). We remark that for all $x \in H_{p,q}$ $\varphi(x, x) = \varphi_J(x)$ and that for all $x \in H_{p,q}$ $\varphi_J\ (Jx)\ = \varphi_J(x)$. We have the following statement :

Theorem 3

The algebra $Cl^{p,q}$ is the real associative algebra generated by the $\varphi_J(x), x \in H_{p,q},\ p \geq 1, q \geq 1$.

Proof

Let us denote by F the real algebra generated by the $\varphi_J(x)$, for all $x \in H_{p,q} \cdot -(H_{p,q}$ is identified with $(E_{2p,2q}\, , J) \cdot -F$ is included into $Cl^{p,q}$. We are going to show that $Cl^{p,q}$ is included into F.

• We notice that for all $x, y \in E,\ \varphi(x, y) \in F$. Then, as for all $x \in H_{p,q},\ \varphi(x, Jx) = \frac{1}{2}(-x^2 + x^2) = 0, 0 \in F$. Moreover, $(\varphi_J(x))^2 = (x \cdot J(x))^2 = -[Q_{2p,2q}, (x)]1 \in R$. As $p > 1$ there exists $x_1 \in E_{2p,2q}$ such that $Q_{2p,2q}(x_1) = 1$ and for $z \in \mathbf{R}_+, \varphi_J\,(\sqrt[4]{z} x_1) \cdot \varphi_J\,(\sqrt[4]{z} x_1)\ \in F$ and, on the other hand $(\varphi_J(\sqrt[4]{z} x_1))^2 = -z$. Thus $-\ z$ and $z \in F$. (We can also use lemma IV.4 p.139 of [6c]. If z is in $\mathbf{R}$ -, - $z = a \in F$ and so $z = -a \in F$. Thus $\mathbf{R} \subset F$.

• We introduce now $C_{2p,2q}\ (s)$, the space called the space of s-vectors and more precisely $C_{2p,2q}(2s)$ and we want to show by a recurrent method that :

$Cl^{p,q} \cap C^+_{2p,2q}(2s) \subset F \cdot C^+_{2p,2q}(2s)$ is the **R**-space generated by 1 and by the products $x_1 \ldots x_{2s}$, where $x_i \in E_{2p,2q}$, for all $i, 1 \leq i \leq 2S$.

• Case of $s = 1$
As **R** is included in F, it is enough to show that for all $x, y \in E_{2p,2q}, x \cdot \tilde{J}(y) + \tilde{J}(x \cdot \tilde{J}(y)) \in F$. Indeed, $z \in Cl^{p,q} \cap C^+_{2p,2q}(2s)$ iff $z = x_1x_2 + \tilde{J}(x_1x_2), x_1x_2 \in E_{2p,2q}$. As $\tilde{J}^2 = -Id$ on $E_{2p,2q}$, there exists $y_2 = \tilde{J}(-x_2) = -\tilde{J}(x_2)$ such that $\tilde{J}(y_2) = x_2$. So z is well of the form: $x \cdot \tilde{J}(y) + \tilde{J}(x \cdot \tilde{J}(y))$.
Moreover, $x \cdot \tilde{J}(y) + \tilde{J}(x \cdot \tilde{J}(y)) = x \cdot \tilde{J}(y) + \tilde{J}(x) \cdot \tilde{J}^2(y) = x \cdot \tilde{J}(y) - \tilde{J}(x) \cdot y$. As, $2R(y, \tilde{J}(x)) = y \cdot \tilde{J}(x) + \tilde{J}(x) \cdot y$ and then -$\tilde{J}(x) \cdot y = -2R(y, \tilde{J}(x) + y \cdot \tilde{J}(x)$.
So, $x \cdot \tilde{J}(y) - \tilde{J}(\cdot x) \cdot y = -2R(y, \tilde{J}(x)) + x \cdot \tilde{J}(y) + y \cdot \tilde{J}(x) = 2\varphi(x, y) - 2R(y, \tilde{J}(x))$ with $2\varphi(x, y) \in F$ and $2R(y, \tilde{J}(x)) \in F$.

• Case for $s > 2$
Let z be in $C^+_{2p,2q}(2s)$. Let us write $z = u \cdot v$, with $u \in C^+_{2p,2q}(2k)$ and $v \in C^+_{2p,2q}(2l)$ with $k, l < s$. By hypothesis, we can assume that $C^+_{2p,2q}(2t) \cap Cl^{p,q} \subset F$ for all $t < s$.
Let us write, now:

$$uv + \tilde{J}(uv) = \left\{ \tfrac{u}{2} + \tilde{J}(\tfrac{u}{2}) \right\} \left\{ v + \tilde{J}(v) \right\} + \left\{ u - \tilde{J}(u) \right\} \left\{ \tfrac{v}{2} - \tilde{J}(\tfrac{v}{2}) \right\} =$$

$$= W_1W_2 + z_1z_2.$$

We easily verify that $\tilde{J}(W_1) = W_1, \tilde{J}(W_2) = W_2$ and that $W_1W_2 \in Cl^{p,q}$. We notice that $\tilde{J}(z_1) = -z_1, \tilde{J}(z_2) = -z_2$ and then, $\tilde{J}(z_1z_2) = z_1z_2$. So, $z_1z_2 \in Cl^{p,q}$. According to the hypothesis of reccurency, W_1W_2 and z_1z_2 belong to F and $uv + \tilde{J}(uv) \in F$.

N.B.: We have found the formula:

for all $x, y \in H_{p,q}, x \cdot J(y) - J(x) \cdot y = 2\varphi(x, y) - 2R(J(x), y)$.

Such a proof naturally leads us to the following definition.

2 – Definition 2 of the Clifford algebra associated with $H_{p,q}$

Let A be an **R**-associative algebra with a unit element.

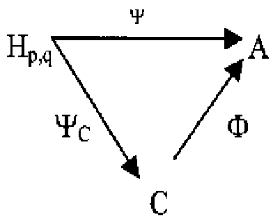
a – Definition of a pseudo-unitary Clifford mapping

We agree to call pseudo-unitary Clifford mapping from $H_{p,q}$ into A any map Ψ from $H_{p,q}$ into A such that
a) $\Psi(\lambda x) = \lambda^2 \Psi(x)$, for all $\lambda \in R$,
b) $\frac{1}{2}\{\Psi(x+y) - \Psi(x) - \Psi(y)\} = \varphi(x,y)$, where λ is an **R**-bilinear mapping from $H_{p,q} x H_{p,q}$ into A,
c) $(\Psi(x))^2 = -[Q_{2p,2q}(x)]^2 1_A$, for all $x \in H_{p,q}$.

We notice that if B is another associative **R** algebra with unit element and if Φ is an homomorphism of algebras with unit elements from A into B, which means that Φ is **R**-linear, multiplicative -($\Phi(aa') = \Phi(a)\Phi(a')$- and that $\Phi(1_A) = 1_B$, then $\Psi_1 = \Phi o \Psi$ from Hp,q into B is a pseudo-unitary Clifford mapping from $H_{p,q}$ into B.
We can easily verify that for all $\lambda \in R, \Psi_1(\lambda x) = \lambda^2 \Psi_1(x)$ and that $\frac{1}{2}\{\Psi_1(x+y) - \Psi_1(x) - \Psi_1(y)\} = \varphi_1(x,y)$ where φ_1 is R-bilinear from $H_{p,q} x H_{p,q}$ into B, and that, for all $x \in H_{p,q}$:
$(\Psi_1(x))^2 = (\Phi o \Psi(x))^2 = \Phi((\Psi(x))^2) = \Phi(-(Q_{2p,2q}(x))^2 1_A) = -Q_{2p,2q}(x))^2 1_B$.

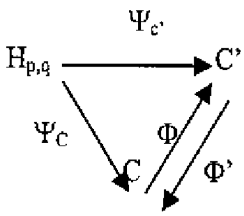
b – Definition 2 of the Clifford algebra associated with $H_{p,q}$

We agree to call Clifford algebra associated with $H_{p,q}$ any R – associative algebra, with unit element 1_C, equipped with a pseudo-unitary Clifford mapping from $H_{p,q}$ into C, which satisfies the following conditions:
1) $\Psi_C(H_{p,q}\ )$ generates C,
2) For any Clifford pseudo-unitary mapping Ψ from $H_{p,q}$ into A, (R-associative algebra with unit element), there exists a homomorphism of algebras with unit elements Φ from C into A, such that
$\Psi = \Phi o \Psi_C$.



$(\forall x)\ (x \in H_{p,q}),\ \Phi\,(\Psi_c\,(x)) = \Psi\,(x)$

The second condition expresses that any pseudo-unitary Clifford mapping of $H_{p,q}$ can be obtained from the map Ψ which is universal. Consequently, if a pseudo-hermitian standard space $H_{p,q}$ possesses a Clifford algebra C, that is unique up to an isomorphism. Indeed, let C' be another Clifford algebra of $H_{p,q}$. The diagram:



where
Φ and Φ' uniquely determined with
$\Phi o\ \Psi_C = \Psi_{C'}$, and $\Phi' o\ \Psi_{C'} = \Psi_C$

implies that : $\Phi' o \Phi o \Psi_C = \Phi' o \Psi_{C'} = \Psi_C$ and $\Phi o \Phi' o \Psi c' = \Phi o \Psi_C = \Psi_{C'}$. As $\Psi_C(H_{p,q})$ generates C and that $\Psi_{C'}(H_{p,q})$ generates C', we can deduce that $\Phi' o \Phi = \mathrm{Id}\ _C$ and $\Phi o \Phi' = \mathrm{Id}\ _{C'} \cdot \Phi$ and Φ' are isomorphisms which are uniquely determined, each of them inverse of the other interchanging Ψ_C in $\Psi_{C'}$ or $\Psi_{C'}$ in Ψ_C.

We can speak of the Clifford algebra of the pseudo-unitary space $H_{p,q}$.

1 – Clifford groups and covering groups of $U(p, q)$

With notations of Deheuvels, [5, b], we introduce the covering groups $R0(2p, 2q)$ and $R0^+(2p, 2q)$ respectively of 0 $(2p, 2q)$ and $S0$ $(2p, 2q)$, associated with the exact sequences:

$1 \to Z_2 \to R0(2p, 2q) \to 0(2p, 2q) \to 1$ and

$1 \to Z_2 \to R0^+(2p, 2q) \to S0(2p, 2q) \to 1$

We introduce Spin $(2p, 2q) = R0^{++}(2p, 2q)$ connected component of the identity in $R0(2p, 2q)$ which is a two-fold covering group of $S0^{+}(2p, 2q) = 0^{++}(2p, 2q)$ associated with the exact sequence:
$1 \rightarrow Z_2 \rightarrow \text{Spin } (2p, 2q) \rightarrow S0^{+}(2p, 2q) \rightarrow 1$
For $p > 1, q > 1 R0(2p, 2q)$ has four connected components by arcs which are two-fold coverings for the corresponding components in $0(2p, 2q)$.
Let $\tilde{G}_{2p,2q}$ be the regular Clifford group constituted by invertible elements g of the Clifford algebra $C_{2p,2q}$ such that for any x in $E_{2p,2q}$, $\Psi(g) \cdot x = (\Pi g)xg^{-1} = y \in E_{2p,2q}$, ($\Pi$ is the principal automorphism of $C_{2p,2q}$). Such group is also the group constituted by products of non isotropic elements in $E_{2p,2q}$, in the Clifford algebra $C_{2p,2q} \cdot \tilde{G}^{+}_{2p,2q}$ denotes the even regular Clifford group: $\tilde{G}^{+}_{2p,2q} = \tilde{G}_{2p,2q} \cap C^{+}_{2p,2q}$. We remark that $\varphi = \Psi$ on $C^{+}_{2p,2q}$.

a - Theorem 4

> For any $v \in U(p, q)$ there exists an invertible element $b_v \in Cl^{p,q}$ determined up to a scalar in R^* such that $\Phi_v(x) = b_v x b_v^{-1} = v(x)$, for all $x \in H_{p,q}$.
> Conversely, for any b invertible belonging to $Cl^{p,q}$ such that for all $x \in H_{p,q}$: $bxb^{-1} = y \in H_{p,q}$, the mapping : $x \mapsto bxb^{-1}$ induces an element of $U(p, q)$.

• The first part is a consequence of Theorem 2 and definition 1 of $Cl^{p,q}$.

• Conversely, with any invertible element b of $Cl^{p,q}$, such that for all $x \in H_{p,q}$, $bxb^{-1} = y \in H_{p,q}$, we can associate $v \in S0(2p, 2q)$ such that $\Phi_v(x) = b_v x b_v^{-1} = v(x)$, for all $x \in H_{p,q}$. We introduce J and $\tilde{J}$ defined as before. Then, for all $x \in H_{p,q}$ and for all $y \in H_{p,q}$, $H_{p,q}$ identified with $(E_{2p,2q}, J)$, such that $Q_{2p,2q}(y) = Q_{2p,2q}(J(y)) \neq 0$, as : $bxb^{-1} \in E_{2p,2q}$, we can write : $\tilde{J}(bxb^{-1}y) = \tilde{J}(bxb^{-1})\tilde{J}(y) = b\tilde{J}(x)b^{-1}\tilde{J}(y)$ id est : $[\tilde{J}(bxb^{-1}) - b\tilde{J}(x)b^{-1}]J(y) = 0$ and according to the hypothesis made for y, we can deduce that for all $x \in E_{2p,2q}$: $\tilde{J}(bxb^{-1}) = b\tilde{J}(x)b^{-1}$, id est $Jov = voJ$ and thus $v \in U(p, q)$ by definition.
Ψ is a natural homomorphism from $\tilde{G}_{2p,2q}$ into $0(2_p, 2_q)$ The restriction of Ψ to $\tilde{G}^{+}(2p, 2q)$ onto $S0(2p, 2q)$ leads us to a surjective homomorphism with kernel R^* associated with the following exact sequence :
$1 \rightarrow R* \rightarrow \tilde{G}^{+}_{2p,2q} \cap Cl^{p,q} \rightarrow U(p, q) \rightarrow 1$.

b – Definition of the pseudo-unitary Clifford group and of covering group $RU(p,q)$ of $U(p,q)$

Definitions

$\tilde{G}^+(2p,2q) \cap Cl^{p,q}$ - determined up to an isomorphism – is called pseudo-unitary Clifford group.
$RU(p,q) = R0^+(2p,2q) \cap Cl^{p,q}$ is called covering group for $U(p,q)$ associated with the exact sequence :
$1 \to Z_2 \to RU(p,q) \to U(p,q) \to 1$

c – Definition of the spinor-group Spin $U_{p,q}$

We recall the exact following sequence of groups
$1 \to Z_2 \to$ Spin $(2p,2q) \to S0^+(2p,2q) \to 1$
Definition

Spin $(2p,2q) \cap Cl^{p,q}$ is called, by definition, spinor group associated with $H_{p,q}$ and is denoted by Spin $U(p,q)$.
We define Ψ (Spin $U_{p,q}) = U_o(p,q)$ as the reduced pseudo-unitary group. We have the following exact sequence

$$1 \to Z_2 \to \text{ Spin } U_{p,q} \to U_o(p,q) \to 1.$$

3 – Fundamental diagram associated with $RU(p,q)$

a – Following a method initiated by Atiyah, Bott and Shapiro [0], we introduce the following definition :

Definition

Let $A(Q)$ be one of the classical group $RO(Q)$ (covering group of $0(Q); G(Q)$ – (Clifford group) – Spin Q. Set $A^u(Q) = A(Q) \times_{z_2} U(1)$ where Z_2 acts on $A(Q)$ and U (1) as $\{\pm 1\}$.

We recall that $U(1)$ is the classical group of complex numbers z with $|z| = 1$ (for the multiplicative law).
We recall the following definition and theorem :

Definition and theorem

Let $(E^cQ^c\ )$ be the complexified of (EQ) where (EQ) is a standard regular quadratic space such that $(Cl(Q))^c \approx Cl(E^cQ^c\ )$.
Let $G^c(EQ)$ the subgroup of invertible elements g of $(Cl(Q))^c$. Which verify : $\forall y \in E, \Pi(g)\gamma g^{-1} \in E$. And let $R0^c(EQ)$ be the kernel of the graduate spinor norm. We have the following exact sequence :
$1 \to U(1) \to R0^c(EQ) \xrightarrow{\delta} 0(Q) \to 1$
and $N(R0^c(E,Q)) = U(1)$

Corollary

We have a natural isomorphism : $R0^c(EQ) \approx R0(EQ) \times_{z_2} U(1)$

b – We recall the following exact sequences
$1 \to Z_2 \to R0(2p, 2q) \to 0(2p, 2q) \to 1$
$1 \to Z_2 \to R0^+(2p, 2q) \to S0(2p, 2q) \to 1$
$1 \to Z_2 \to RU(p, q) \overset{\varphi=\Psi}{\to} U(p, q) \to 1$

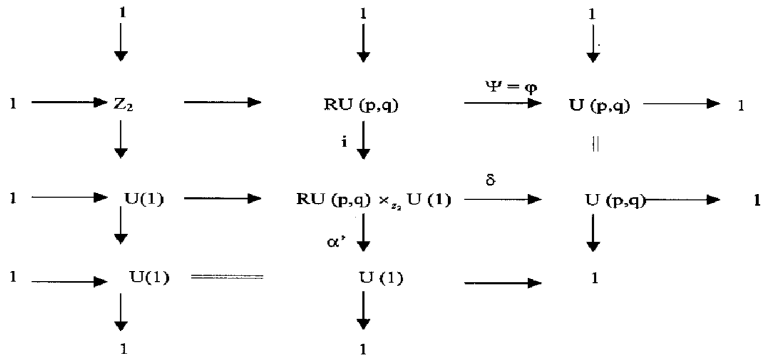
Definitions

Let us introduce : $\alpha\ : z \mapsto \alpha(z) = z^2$ from $U\ (1)$ into $U\ (1)$
α' : $[v, u] \in RU(p, q) \times_{z_2} U(1) \to \alpha'\ ([v, u]) = u^2 \in U(1)$
where $[v, u]$ denotes the class of $(v, u) \in RU(p, q) \times_{z_2} U(1)$
$\delta\ : \delta[g, z] = \Psi(g)$
$i\ : i(g) = [g, 1]$ for all $g \in RU(p, q)$ and all $z \in U(1)$

We have the following statement :

b – Proposition

We have the following commutative diagram of Lie groups associated with $RU(p, q)$



5 – Characterization of $U(p,q)$

a – Let us assume that $p+q=n=2r$

We consider the following basis of $_R(C^n)$: $\{e_1,\ldots,e_n,Je_1,-,Je_n\}$ and we introduce, as previously, $Q_{2p,2q}$ – (of signature $(2_p,2_q)$) – quadratic form associated with the bilinear real symmetric form R. Let $E_{2n}=\mathbf{R}^{2n}$ and let $E_{2n}^{'}$ be the complexified of E_{2n}, $2n$ – dimensional C-space – $(2n=4r)$.
We know that there exists a special Witt – decomposition of $E_{2n}^{'}$, $E_{2n}^{'}=F\oplus F'$, where F, respectively F', is a maximal totally isotropic $2r$-dimensional subspace.
We write $F=\{x_1,\ldots,x_n\}, F'=\{y_1,\ldots,y_x\}$ with respective explicit basis :

$$x_1=\frac{e_1+e_n}{2},\ldots,x_p=\frac{e_p+e_{n-p+1}}{2},x_{p+1}=\frac{ie_{p+1}+e_{n-p}}{2},\ldots,x_r=\frac{ie_r+e_{n-r+1}}{2},$$

$$x_{r+1}==\frac{J(e_1)+iJ(e_n)}{2},\ldots,x_{r+p}=\frac{J(e_1)+J(e_{n-p+1})}{2},x_{r+p+1}=\frac{iJ(e_{p+1})+J(e_{n-p})}{2},\ldots,$$

$$x_n=\frac{iJ(e_r)+J(e_{n-r+1})}{2},$$

$$y_1=\frac{e_1-e_n}{2},\ldots,y_p=\frac{e_p-e_{n-p+1}}{2},y_{p+1}=\frac{ie_{p+1}-e_{n-p}}{2},\ldots,y_r=\frac{ie_r-e_{n-r+1}}{2},$$

$$y_{r+1}==\frac{J(e_1)-J(e_n)}{2},\ldots,y_{r+p}=\frac{J(e_p)-J(e_{n-p+1})}{2},\ldots,$$

$$y_{r+p+1}=\frac{iJ(e_{p+1})-J(e_{n-p})}{2},\ldots,y_n=\frac{iJ(e_n)-J(e_{n-r+1})}{2}$$

with for $1 \leq j \leq p, \bar{x}_j =, x_j \bar{y}_j = y_j$, for $r+1 \leq j \leq r+p \bar{x}_j = x_j, \bar{y}_j = y_j$ and for $p+1 \leq j \leq r, \bar{y}_j = -x_j$ and for $r+p+1 \leq j \leq n, \bar{y}_j = -\bar{x}_j$ and with $B(x_i, y_j) = \frac{\delta_{1\gamma}}{2}$, $B(x_i, x_j) = B(y_i, y_J) = 0$ and thus $x_i y_j + y_j x i = \delta_{ij}, 1 \leq i, j \leq n$.
We recall that $J|F = i$ Id and that $J|F' = -i$ Id.

b – Characterization of $U(p,q)(p+q = n = 2r, p \leq q, p \leq r)$.

Let us consider $Cl'(2_p, 2_q)$ the complexified of $Cl(2_p, 2_q)$. As usually, we define $\exp(\lambda X)$, $\lambda \in C$, for $X \in Cl'(2_p, 2_q)$.
We know that if $XY = YX$, then $\exp X \exp Y = (X+Y), (\exp X)^{-1} = \exp(-X)$, where $\exp X = \sum\limits_{k \geq 0}^{\frac{X^k}{k!}}$ and that $\tau(\exp X) = \exp(\tau(X))$, where τ denotes the principal anti-automorphism of the Clifford algebra.
We recall that $U(p,q)$ is the set of elemets u of $S0(2p, 2q)$ such that $uoJ = Jou$.
We want to prove the following statement – $(i \in C\ : i^2 = -1, t \in \mathbf{R})$
(We denote by Ψ the classical projection already considered : $\Psi(g) \cdot x = \Pi(g) \cdot x^{-1})$.
Proposition 1

$\Psi \exp[it \sum\limits_{k=1}^{n} (x_k y_k)]$ induces the following mappings $x \mapsto e^{it} x$ on F and $x \mapsto e^{-it} x$ on F'.

Proof

Lemma 1
$(x_k y_k)(x_l y_l) = (x_l y_l)(x_k y_k)$ and $\exp(it\ x_k y_k) \exp(it\ x_l y_l) = \exp(it\ (x_k y_k + x_l y_l))$.
The result is quite straightforward.

Lemma 2

Let $z = \exp(it\ [\sum\limits_{k=1}^{n} (x_k y_k)] = \prod\limits_{k=1}^{n} \exp[it\ x_k y_k]$, then $N(z) = e^{int} = (e^{it})^n$ and $N(z) = 1$.
Let us consider, now, the plane generated by x_1 and y_1 such that $x_1^2 = 0, y^2 = 0$ and $2R(x,y) = 1$. It is easy to verify that $\tau(\exp it\ x_1 y_1) = \tau(z_1) = \exp(it\ y_1 x_1)$ and that $N(z) = e^{it}$, thus $|N(z)| = 1$ and $z_1^{-1} = e^{-it} \exp(it\ y_1 x_1)$

Then it is easy to verify that $e^{-it}\exp(it\ y_1x_1)x_1\exp(it\ y_1x_1) = e^{it}x_1$ and that $e^{-it}\exp(it\ x_1y_1)y_1\exp(it\ y_1x_1) = e^{-it}y_1$.
The result is quite obvious by recurrency.

Corollary

$$\Psi o \exp\left\{i\pi\left(\sum_{k=1}^{n} x_k y_k\right)\right\} = -Id \text{ on } (R_{2n})'$$
$$\Psi o \exp\left\{2i\pi\left(\sum_{k=1}^{n} x_k y_k\right)\right\} = Id \text{ on } (R_{2n})'$$
$$\Psi o \exp\left\{i\frac{\pi}{2}\left(\sum_{k=1}^{n} x_k y_k\right)\right\} = J$$

Proposition 2

The group $U(p,q)$ is identical with $\Psi(\Delta U_{p,q})$ where $\Delta U_{p,q}$ is the set of products of elements z with $|N(z)| = 1$ such that $z = \exp(i\lambda)\exp(ia^{kl}x_ky_l)$, (with summation in k and l) , $a^{kl} \in C$, with $a^{kl} = \bar{a}^{lk}$ and $\lambda = -\frac{\sum a^{kk}}{2}$.

Proof

• First, it is easy to verify that $U(p,q)$ is included into $\Psi(\Delta U_{p,q})$. It is enough to notice that x_ky_k commutes with $\sum_l x_ly_l$ and to use the previous corollary, to express a condition of reality, using the fact that for $z = \exp(i\lambda)\exp(ia^{kl}x_ky_l) = \mu\exp(u)$, with $\mu = \exp(i\lambda)$ and $u = ia^{kl}x_ky_l$; $N(z) = \mu^2\exp(u+\tau(u)) = \mu^2\exp(ia^{kl}(x_ky_l + y_lx_k)) = \mu^2\exp[ia^{kl}\delta_{kl}] = \exp(2i\lambda)\cdot\exp(i\sum_k a^{kk})$.

• Then we notice that $U(p,q)$ is compact in $GL(n,C)$. Consider $\Psi o\exp(it\ x_ky_k)$. The tangent map to identity is surjective on the set of x_ky_l which generates a subspace of dimension n^2 with $n^2 = \dim U(p,q)$. The result is obtained by considering the value of the norm and the fact that the exponential map is surjective on any compact connected Lie group.

Remarks

1 – Previously, we assumed that $n = p+q = 2r$. If $n = 2r+1$ then $2n = 2r+2$ is even and we can consider a special Witt decomposition of $E'_{2n}(2p,2q)$ which leads to the same conclusions.

2 - We notice the following exact sequence
$1 \to U(1) \to \Delta U(p,q) \to U(p,q) \to 1$

So $\Delta U(p,q)$ is isomorphic with $RU(p,q)\times_{z_2}U(1)$, which gives an algebraic characterization of $RU(p,q)\times_{z_2}U(1)$.

6 – Associated spinors

First, we recall the following classical results [5b, p.331]

a – Let (E,q) be a quadratic regular complex space.
If $\dim E=2k$, the Clifford algebra $C(E,Q)$ is isomorphic with $m(2^k,\ \mathbf{C})$
If $\dim E=2k+1$, the Clifford algebra $C(E,Q)$ is isomorphic with $m(2^k,\ \mathbf{C})\oplus m(2^k,\ \mathbf{C})$

b – We have introduced $E_{2n}=R^{2n}$ endowed with a quadratic form of signature $(2p,2q)$ and the complexified E'_{2n}, $2n$ – dimensional complex space with its own Clifford algebra isomorphic with the complexified of the Clifford algrebra of E_{2n}.
Inside this Clifford algebra $[C(R^{2n},\ Q_{2p,2q})]$' we have considered the group $\Delta U(p,q)\approx RU(p,q)\times_{z_2}U(1)$, associated with the exact sequence :
$1\rightarrow U(1)\rightarrow\Delta U(p,q)\rightarrow U(p,q)\rightarrow 1$
According to the previous result as $\dim\limits_{C} E'_{2n}=2n$, the Clifford associated algebra A is isomophic with $m(2^n,\ C)$. A is identical with $\underset{C}{L}(S)$ where S is a complex 2^n – dimensional space, minimal module, [4, 5b] of such an algebra $A\cdot A$ is a central simple complex algebra.

Definition

S is by definition the space of spinors associated with such an algebra : $\dim\limits_{C} S=2^n$

c – Pseudo-hermitian structure on S

A. Weil has shown, [38], that for an antilinear involution α over A, a central simple complex algebra, if we denote by l(a) the endomorphism $x\mapsto ax$ of the underlying vecto-space to A and if we consider the trace $Trl(a)$, the form $(x,y)\in A\mapsto Trl(x^\alpha y)$ is a non degenerate hermitian form associated with the antilinear involution α. R. Deheuvels has shown in [5, b] that α determines on S a pseudo-hermitian scalar product for which α is precisely the operator of adjunction. Moreover, Deheuvels proved that the signature of the corresponding quadratic hermitian form-(associated with $(x,y)\mapsto Trl(x^\alpha y)$) - is $(r^2+s^2,2rs)$. Let us choose now for $\alpha:\tau$ the principal anti-automorphism of the Clifford algebra A, central simple complex algebra for which τ is antilinear. Let us take again the proof given in [5, b]. It is easy to see that the

pseudo-hermitian form is a neutral one, $r^2 + s^2 = 2rs$ id est $r = S$. So, the pseudo-hermitian scalar product on S is neutral of signature $(2^{n-1}, 2^{n-1})$.
The pseudo-unitary group of automorphisms of S which conserve such a scalar product is constituted by elements u of $\underset{C}{L}(S) \approx A \approx m(2^n, C)$ such that $u^\tau u = 1$.
After imbedding of $RU(p,q)$ into the complexified algebra A by the canonical injection we obtain that $RU(p,q)$ is contained into $U(2^{n-1},\ 2^{n-1})$. We want to show that for $p \geq 2$, Spin $U(p,q)$ is, in fact, contained into $SU(2^{n-1}, 2^{n-1})$.
Proof:

Any element $g \in$ Spin $U(p,q)$ is the product of an even number of vectors u_i such that $N(u_i) = 1$ and of an even number of vectors u_j, such that $N(u_j) = -1, g = u_1 u_2 \dots u_{2k}$.
As $u_1 u_2 = u_2(u_2^{-1} u_1 u_2)$ and as $y_1 = u_2^{-1} u_1 u_2 \in E_{2n}$ with $N(y_1) = N(u_1)$, we can assume that the u_i with $N(u_i) = -1$, if they exist, are set before in the writing of g. Moreover, if two "u_i" are linearly dependent, using previous permutations, we are led to a factor $\pm$ 1. So, we can assume that $g = u_1 \dots u_{2k}$ with u_i linearly independent, two by two, with $N(u_i) = -1$ before, if they exist.
If (u_i) verify $(u_i^2) = 1 = N(u_i), u_i$ is an involutive operator of S and so its determinant equals ± 1. Let us consider two consecutive vectors u_1, u_2, linearly independent with $N(u_1) = N(u_2) = -1$ and let P be the plane which they generate. If $p \geq 1$ – (in fact $2p \geq 2$) – there exists $z \in E_{2n}$ such that $R(z,z) = 1, R(z,u_1) = R(z,u_2) = 0$ and $(zu_1)^2 = 1, (zu_2)^2 = 1\, zu_1 zu_2 = -u_1 u_2$. So, zu_1 as zu_2 is an involutive operator of S, with determinant equal to ± 1. (cf. appendix).
Thus, any $g \in$ Spin $U(p,q)$ is the product of elements which have a determinant equal to ± 1. So, Spin $U(p,q)$ is contained in the subgroup of the pseudo-unitary group constituted by elements of determinant ± 1, but as Spin $U(p,q)$ is connected, all these elements have + 1 as determinant. We have obtained the following theorem.

Theorem

The space S of spinors associated with A inherits a natural complex structure and a pseudo-unitary neutral scalar product of signature $(2^{n-1},\ 2^{n-1})$, up to a scalar factor, which is conserved by the group Spin $U(p,q)$. We have the following imbedding : Spin $U(p,q)$ is contained into $SU(2^{n-1}, 2^{n-1})$.

7. Natural Imbeddings of the Projective Quadrics $\tilde{H}_{p,q}$

The imbedding can be made as in [5, b].
Let S be the space of spinors previously introduced. Let [|] be a scalar product on S associated with the involution τ, id est, for $a \in Cl^{p,q}$, a and a^τ linear operators of S are adjuncts of each other relatively to the scalar product [|].
The injective mapping :
$\{$isotropic line $\{\lambda x\ \}$ in $H_{p,q}\} \rightarrow \{$maximal totally isotropic subspace $S(x) = Im(xy)_S = ker(yx)_s\}$ where $(xy)_s$ and $yx)_s$ are the projectors of S defined by the elements xy and yx of $Cl^{p,q}$, determines a natural imbedding of the projective quadric $\tilde{H}_{p,q}$ into the grasmannian of half dimensional subspaces $G(S, \frac{1}{2}\dim S)$.
According to general results of [28] (Th. 12-19, p.237 ; Prop. 17-46 p.358] we obtain that $\tilde{H}_{p,q}$ is homeomorphic with $U(2^{n-1})$. Then we have the following summary :

Theorem

Spin $U(p,q) \subset U(2^{n-1}, 2^{n-1})$; $\tilde{H}_{p,q}$ is homeomorphic with $U(2^{n-1})$.

Appendix

Proof of the lemma used in Chapter 6.– 6. $E_{r,s}$ denotes R^{r+s} endowed with a quadratic form q of signature (r,s) ; $R(x,y)$ denotes the symmetric bilinear associated form.

Lemma

Let (V, q) the quadratic standard pseudo-euclean space $E_{r,s}$ with $r \geq 2$. For any pair of vectors $\{u_1, u_2\}$linearly independent such that $R(u_1, u_1) = R(u_2, u_2) = -1$, there exists $z \in V$ such that $R(z, z) = 1$ and $R(z, u_1) = R(z, u_2) = 0$.

A quadratic real plane P which inherits two linearly independent vectors u_1 and u_2 with $q(u_1) = q(u_2) = -1$ is necessarily isomorphic with one of the three following standard planes (R^2, q) :

$$P_1 \ : q_1(x) = -(x_1)^2 \ ; E_{o,2} \ : q_2(x) = -(x^1)^2 - (x^2)^2 \ ;$$

$$E_{1,1} \ : q_3(x) = (x^1)^2 - (x^2)^2.$$

In the last two cases P is a direct factor of $E_{r,s}$ and $E_{r,s} = P \oplus P^\perp$. According to the classical Witt-isomorphism theorem [4 ; 5 b] ; we have :
- if $P \approx E_{0,2}, P^\perp \approx E_{r,s-2}$ with $r \geq 1$
- if $P \approx E_{1,1}, P^\perp \approx E_{r-1,s-1}$. In the first case and in the second one, if $r \geq 2$, there always exists $z \in P^\perp$ with $q(z) = 1$,
- if $P \approx P_1$ and if D is the isotropic line of P, there exists an isotropic line D', linearly independent of P such that $D \perp D'$ and so $P \oplus D'$ is regular and isomorphic with $E_{1,2}$. According to Witt-Theorem, $(P \oplus D')^\perp \approx E_{r-1,s-2}$. We obtain the existence of z, if $r \geq 2$.

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