BETWEEN QUANTUM VIRASORO ALGEBRA $\mathcal{L}_c$ AND GENERALIZED CLIFFORD ALGEBRAS

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Abstract. In this paper we construct the quantum Virasoro algebra $\mathcal{L}_c$ generators in terms of operators of the generalized Clifford algebras C_n^k. Precisely, we show that $\mathcal{L}_c$ can be embedded into generalized Clifford algebras.

1. Introduction

Over the past decade, much attention has been paid to the generalized Clifford algebras and its associated Grassmann algebras in connection with the important role investigated in mathematics and physics (see e.g [1, 2]). In fact, the linearization of the homogeneous polynomials of degree n and k variables leads to generalized Clifford algebras C_n^k and to the introduction of the multicomplex plane $\mathcal{MC}_n$ where the fundamental element e satisfies the basic relation $e^n = -1$ [3]. On the other hand, the generalized Grassmann algebras which have been introduced firstly in framework of the 2D conformal field theories and they arise also in the contexts of quantum groups and fractional generalization of supersymmetry quantum mechanics [4] and next they were used for generalization of supersymmetry to the fractional supersymmetry [5] which have found applications to various physical problems, such as the description

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of the fractional statistics (see e.g [6])

In this letter, we purpose the construction of the quantum conformal symmetry $\mathcal{L}_c$ in terms of the generalized Clifford generators γ_i. In fact, in a previous paper [7], we have done the embedding of the classical Virasoro algebra $\mathcal{L}_0$ (with out central charge c) into generalized Clifford algebras here we will extend this embedding to the case of quantum Virasoro algebra $\mathcal{L}_c$ (with central charge c). First, we will recall some notions connected with the generalized Clifford algebras and its associated generalized Grassmann algebras. Then, after recalling the classical and quantum conformal symmetries, we will present our embedding of $\mathcal{L}_c$ algebra into C_n^k. In fact, we will show that the generators of $\mathcal{L}_c$ may be expressed via the generators γ_i of the generalized Clifford algebras.

2. Preliminaries on the Generalized Clifford Algebras

In brief, the n-dimensional generalized Clifford algebras of order k (GCA) is defined as an associative algebra over complex numbers $\mathcal{C}$, generated by operators (matrices) γ_i (for more details see [1, 2]) satisfying the following commutation relations

$$\gamma_i\gamma_j = \omega\gamma_j\gamma_i, \quad i < j \quad (i, j = 1, 2, ...n) \tag{1}$$

and

$$(\gamma_i)^k = 1 \quad i = 1, 2, ...n \tag{2}$$

where $\omega = e^{\frac{2\pi i}{k}}$ is a kth primitive root of unity.

The generators $\{\gamma_i\}_1^n$ can be represented as tensor product of the generalized Pauli matrices [1, 2] in complete analogy with the usual Clifford algebra. If we substitute the equation (2) by $(\gamma_i)^k = 0$, we obtain the so-called generalized Grassmann algebras which have been introduced firstly in framework of the 2D conformal field theories and were rediscovered in the contexts of quantum groups and fractional generalization of susy quantum mechanics and next these algebras were used also for generalization of supersymmetry to the fractional supersymmetry.

3. Classical and Quantum Virasoro Symmetries

For reader convenience we remind some information about classical and quantum Virasoro symmetries termed also conformal symmetries, these latter were first introduced in the context of string theories and they are relevant to any theory in (2+1)-dimensional space-time which possesses conformal invariance. The conformal symmetry consists of all general transformations

$$z \to z + \epsilon(z), \qquad \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) \tag{3}$$

where $\epsilon(z)$ and $(\bar{\epsilon}(\bar{z}))$ are an infinitesimal analytical (anti-analytical) functions. It can be represented as an infinite Laurent series

$$\epsilon(z) = \sum_n \epsilon_n z^{n+1}, \qquad n \in \mathcal{Z}, \tag{4}$$

and an analogous formula holds for $\bar{\epsilon}(\bar{z})$.

These mappings are generated by the differential operators

$$V_m = -z^{m+1}\partial_z \quad \text{and} \quad \bar{V}_m = -\bar{z}^{m+1}\partial_{\bar{z}}. \tag{5}$$

The operators ℓ_m satisfy the commutator relations

$$[\, V_m, V_n] = (m-n)V_{m+n}, \tag{6}$$

and an analogous formula holds for the $\bar{V}_m$. Hence, this means that both the V_m and the $\bar{V}_m$ span an infinite-dimensional Lie algebra. Moreover, these two algebras are combined as a direct sum, $[V_m, \bar{V}_m] = 0$. The algebra defined by (6) is known as the classical conformal or classical Virasoro algebra. We shall denote that the quantum version of the above algebra is obtained by adding the central terms which is connected with the quantum theory. Then, it is well-known that the classical Virasoro admits a unique 1-dimentional central extension

$$\mathcal{L}_c = \mathcal{L} \oplus \mathcal{C}c, \tag{7}$$

with the following commutation relations

$$[\, V_m, c] = 0, \qquad [V_m, V_n] = (m-n)V_{m+n} + c\frac{(m^3 - m)}{12}\delta_{m+n,0} \tag{8}$$

where the value of the central charge c is the parameter of the theory in the quantum field theory context (e.g $c = 1$ for the free boson). The unitary representation of the above algebra is well-know (see e.g [8]).

4. $\mathcal{L}_c$ as a Subalgebra of Generalized Clifford Algebras

First, recall that in a previous paper [8], we have realized the centerless $\mathcal{L}_0$ Virasoro algebra (alias classical Virasoro algebra) as subalgebra of the generalized Clifford algebras as a linear combination of the generators γ_i. In fact for any pair (i,j) such that $i<j$ we have :

$$V_N^{(i,j)} = \sum_{\bar{m}\neq(k,k),\neq(0,0)} D_N^{\bar{m}} \gamma_i^{m_1}\gamma_j^{m_2}, \tag{9}$$

where $\bar{m} = (m_1, m_2)$, and generators $\gamma_i^{m_1}\gamma_j^{m_2} \sim e^{\bar{m}}_{(i,j)}$ satisfy the so-called centerless area preserving algebras (for k large and after a renormalization of generators)

$$[e^{\bar{m}}_{(i,j)}, e^{\bar{n}}_{(i,j)}] = (\bar{m}\times\bar{n}) e^{\bar{m}+\bar{n}}_{(i,j)}, \tag{10}$$

where $\bar{m}\times\bar{n} = m_1 n_2 - m_2 n_1$. Imposing that the generators $V_N^{(i,j)}$ satisfying

$$[V_N^{(i,j)}, V_{N'}^{(i,j)}] = (N-N')V_{N+N'}^{(i,j)}. \tag{11}$$

Then, this is equivalent to the following identities among the constants $D_N^{\bar{m}}$ namely

$$\sum_{\bar{m}\neq(k,k),\neq(0,0)} D_{N'}^{\bar{l}-\bar{m}} D_N^{\bar{m}} (\bar{m}\times\bar{l}) = (N-N')D^{\bar{l}}_{N+N'} \quad (\bar{l}\neq(0,0)). \tag{12}$$

The particular simple solution is chosen to be of the forme

$$D_N^{\bar{m}} = \frac{(-1)^{m_1}}{m_1}\delta_{m_2,N} \qquad m_1 \neq 0 \tag{13}$$

$$D_N^{\bar{m}} = 0 : \qquad\qquad m_1 = 0. \tag{14}$$

In what follows, we will extend the work done below on $\mathcal{L}_c$ Virasoro algebra (alias quantum Virasoro algebra eq (8)). Indeed, if we now impose that the generators $e^{\bar{m}}_{(i,j)}$ satisfy the area preserving with cental non-trivial charge

$$[e^{\bar{m}}_{(i,j)}, e^{\bar{n}}_{(i,j)}] = (\bar{m}\times\bar{n}) e^{\bar{m}+\bar{n}}_{(i,j)} + \bar{a}.\bar{m}\delta_{\bar{m},\bar{n}}, \tag{15}$$

where $\bar{a}.\bar{m} = a_1 m_1 + a_2 m_2$. Then, the linear combination eq (9) satisfies a quantum Virasoro $\mathcal{L}_c$ algebra relation eq (8), in fact

$$[V_N^{(i,j)}, V_{N'}^{(i,j)}] = (N-N')V_{N+N'}^{(i,j)} + D_{N,N'} \tag{16}$$

with $D_{N,N'}$ is given by

$$D_{N,N'} = \sum_{\bar{m}\neq(0,0)} D_N^{\bar{m}} D_{N'}^{-\bar{m}} \bar{a}.\bar{m} \tag{17}$$

From the Jacobi identity for the $e^{\bar{n}}_{(i,j)}$'s

$$[e^{\bar{m}}_{(i,j)}, [e^{\bar{n}}_{(i,j)}, e^{\bar{l}}_{(i,j)}]] + \text{cyclic perm.} = 0 \tag{18}$$

the constant $D_{N,N'}$ are consistent with the Jacobi identity for $V_N^{(i,j)}$

$$[V_M^{(i,j)}, [V_N^{(i,j)}, V_L^{(i,j)}]] + \text{cyclic perm.} = 0 \tag{19}$$

Which imply that

$$(M-N)D_{L,N+M} + (M-L)D_{M,N+L} + (L-M)D_{N,L+M} = 0 \tag{20}$$

and, additionally, we have also the skew-symmetric relation

$$D_{M,N} = -D_{N,M} \tag{21}$$

Following [9], using the transformation $V_M^{(i,j)} \to V_M^{(i,j)} + MV_M^{(i,j)}D_{M,0}$, where $M \neq 0$ and $V_0^{(i,j)} \to V_0^{(i,j)} + \frac{1}{2}D_{1,-1}$, we obtain afterwards $D_{M,0} = D_{0,M} = D_{1,-1} = 0$. Then putting $L = 0$, we deduce that $D_{M,N} = 0$ unless $M = -N$ and putting $L = -M - 1$ and $N = 1$ in the Jacobi identity. We finely find that the most general solution of the eq (20) is the form

$$D_{M,N} = \frac{d}{12} M(M^2-1)\delta_{M+N=0}, \tag{22}$$

where d is some constant.

Hence, by imposing that the generators $\gamma_i^{m_1}\gamma_j^{m_2} \sim e^{\bar{m}}_{(i,j)}$ satisfying the area preserving algebra with non-trivial charge, we have see that the embedding of the classical Virasoro algebra into generalized Clifford algebras can be extended to the quantum Virasoro algebra. Finally, it is interesting to modify the construction giving the w_∞-algebra in terms of the generators γ_i algebra and leading to the central extensions.

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