The Twistor Structure of the Biquaternionic Projective Point

Alfonso F. Agnew
Department of Mathematics
California State University at Fullerton
Fullerton, CA, U.S.A.
aagnew@fullerton.edu

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Abstract. Souček [1, 2] discovered an intriguing connection between the standard twistor correspondence and the biquaternionic projective line $\mathbb{BP}^1$. The biquaternionic projective point, $\mathbb{BP}^0$, also has twistor structure corresponding to the collection of α- or β-planes passing through the origin in spacetime. The duality between α- or β-planes is shown to correspond to the choice of left vs. right scalar action. Moreover, we find that $\mathbb{BP}^0$ is homeomorphic to the scheme $\mathbb{CP}^1$.

1. Introduction

The biquaternion algebra $\mathbb{B}$ is non-commutative and contains non-invertible elements corresponding to the zero divisors of the algebra. In particular, it is not a field. Therefore, when building a projective space over $\mathbb{B}$, the standard (i.e. field case) construction of taking a free vector space V^n modulo nonzero field elements must be generalized. One method is take a free module over the algebra and take the quotient with respect to the invertible elements of the algebra. In the case of an algebra with zero divisors, such as $\mathbb{B}$, one may obtain a non-trivial "projective point". $\mathbb{BP}^0$ contains one more than a sphere's worth of equivalence classes [1]. Moreover, using the natural quotient topology, one finds that $\mathbb{BP}^0$, has the topological structure of the 2-sphere with an additional, dense, point p. This is precisely the topological structure of the scheme-theoretic complex projective space of one-dimension. We address these issues in §2.

In [1, 2], Souček shows how, remarkably, the standard twistor correspon-

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dence is encoded in the non-Hausdorff character of the 1-dimensional biquaternionic projective space $\mathbb{BP}^1$. In §3, we show that, as a right projective space, $\mathbb{BP}^0$ also has twistor structure. In particular, we find that $\mathbb{BP}^0 - \{p\}$ corresponds to the collection of $\beta-$planes passing through the origin in complex Minkowski space $\mathbb{CM}$. We then compare the right and left projective points to find that, although they are homeomorphic, their twistor interpretations embody the duality between $\alpha-$ and $\beta-$ planes. In §4 we summarize and offer concluding remarks and discuss current and future research issues.

2. $\mathbb{BP}^0$ and its Topology

2.1. Biquaternions

In short, the biquaternions are the tensor product $\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$. More constructively, we denote $1 = i_0$ and the standard quaternionic units by i_1, i_2, i_3. The quaternionic units satisfy

$$i_l^2 = -1, \quad l \in \{1,2,3\}, \tag{1}$$

$$i_1 i_2 = i_3 \quad \text{(and cyclic permutations of 1,2,3)}. \tag{2}$$

The *complex* algebra $\mathbb{B}$ is generated by $\{i_\mu\}, \mu = 1,2,3,4$. Therefore, a typical element has the form

$$q = q^\mu i_\mu := \sum_{\mu=0}^{3} q^\mu i_\mu = q^0 + q^1 i_1 + q^2 i_2 + q^3 i_3, \tag{3}$$

where $q^\mu = x^\mu + iy^\mu \in \mathbb{C}$ for each $\mu = 0,1,2,3$. Lastly, i (hence all q^μ), commute with each of the quaternionic units i_μ.

The square norm

$$|q|^2 = qq^\dagger = (q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 \tag{4}$$

is induced by the conjugation

$$q^\dagger = q^0 - q^1 i_1 - q^2 i_2 - q^3 i_3. \tag{5}$$

The biquaternions are isomorphic, as a pseudo-normed space, to four–dimensional complex Minkowski space, $\mathbb{CM}$, as seen by the map

$$\mathbb{CM} \ni z^\mu := (z^0, z^1, z^2, z^3) \overset{\iota}{\mapsto} z^0 + (-iz^1)i_1 + (-iz^2)i_2 + (-iz^3)i_3 =: z \in \mathbb{B},$$

whence,

$$\begin{aligned} |z|^2 &= (z^0 - iz^1 i_1 - iz^2 i_2 - iz^3 i_3)(z^0 - iz^1 i_1 - iz^2 i_2 - iz^3 i_3)^\dagger & (6)\\ &= (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 & (7)\\ &= \eta_{\mu\nu} z^\mu z^\nu. & (8) \end{aligned}$$

A biquaternion q has an inverse if and only if $|q|^2 \neq 0$, and in this case,

$$q^{-1} := \frac{q^\dagger}{|q|^2}.$$

Furthermore, q is a zero divisor of the algebra if and only if $|q|^2 = 0$.

Since the biquaternionic norm can be viewed as resulting from the multiplication and conjugation within the *algebra* $\mathbb{B}$, we get an algebraic characterization of the "light-like" biquaternions $\mathcal{N}$ as the set of zero divisors of the algebra. We use the terminology "light-like" biquaternions for those biquaternions with zero norm, which correspond (via ι) to the light-like vectors in $\mathbb{CM}$. The group of invertible elements of the algebra is then $\mathbb{B}^* = \mathbb{B} - \mathcal{N}$, which correspond to spacelike and timelike vectors.

$\mathbb{B}$ is also isomorphic (as algebras over $\mathbb{R}$ or $\mathbb{C}$) to $\mathbb{C}^{2\times 2}$. This is particularly convenient, as matrices are very familiar objects, and as such, we freely use these standard isomorphisms:

$$\mathbb{B} \xrightarrow{\ \iota_1\ } \mathbb{C}^{2\times 2} \xleftarrow{\ \iota_2\ } \mathbb{CM}$$

Explicitly,

$$\begin{aligned} \iota_1(q^\mu i_\mu) &:= \begin{pmatrix} q^0 + iq^3 & -q^2 + iq^1 \\ q^2 + iq^1 & q^0 - iq^3 \end{pmatrix} & (9)\\ \iota_2(z^\mu) &:= \begin{pmatrix} z^0 + z^3 & z^1 + iz^2 \\ z^1 - iz^2 & z^0 - z^3 \end{pmatrix} & (10) \end{aligned}$$

Moreover, the norms correspond to the determinant. Then, $\mathbb{B}^*$ corresponds to $GL(2,\mathbb{C})$, and $\mathcal{N}$ corresponds to $\mathbb{C}^{2\times 2} - GL(2,\mathbb{C})$.

2.2. $\mathbb{BP}^0$

2.2.1. Construction of $\mathbb{BP}^0$

Recall, for example, how one performs the standard construction of complex projective n-space, $\mathbb{CP}^n$. One first forms the free complex vector space of dimension $n+1$, $\mathbb{C}^{n+1}$, then takes the quotient of $\mathbb{C}^{n+1} - \{\vec{0}\}$ modulo the equivalence relation

$$\vec{u} \sim \vec{v} \iff \vec{u} = \lambda\vec{v}, 0 \neq \lambda \in \mathbb{C}^*, \tag{11}$$

where $\mathbb{C}^*$ denotes the invertible (i.e. nonzero) complex numbers. Then, $\mathbb{CP}^n := (\mathbb{C}^{n+1} - \{\vec{0}\})/\sim$.

Unlike $\mathbb{C}$, $\mathbb{B}$ is not a field; it is not commutative and contains non-zero, non-invertible elements. Thus, in order to construct projective spaces over $\mathbb{B}$, the above procedure must be generalized. As it turns out[1], there are many inequivalent ways to generalize the above procedure depending on the desired properties of the resulting projective space. For an example from chain geometry that differs from what follows, see [3]. Following Souček, we choose the following generalization. In analogy with the complex case, we form the free, right $\mathbb{B}$-module $\mathbb{B}^{n+1}$, and take the quotient modulo the equivalence relation

$$u \sim v \iff u = v\lambda, \lambda \in \mathbb{B}^*, \tag{12}$$

where $\mathbb{B}^*$ denotes the invertible (i.e. nonzero-normed) biquaternions. The space $\mathbb{BP}^n$ is then defined to be the quotient space $(\mathbb{B}^{n+1} - \{\vec{0}\})/\sim$.

2.2.2. Topology of $\mathbb{BP}^0$

Consider the isomorphism $\mathbb{B} \leftrightarrow \mathbb{C}^{2\times 2}$. In matrix terms, then, $\mathbb{BP}^0$ is the space of orbits of $GL(2,\mathbb{C})$ acting by right multiplication on $\mathbb{C}^{2\times 2} - \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. One orbit p consists of $GL(2,\mathbb{C})$ itself. The remaining orbits consist of subsets of singular matrices. Therefore, representatives of these orbits will have the form

$$Q = \left(\vec{Q}^0 \ \vec{Q}^1 \right) = \left(\vec{Q}^0 \ \beta\vec{Q}^0 \right) \tag{13}$$

if $\vec{Q}^0 \neq \vec{0}$, else

$$Q = \left(\vec{Q}^0 \ \vec{Q}^1 \right) = \left(\vec{0} \ \vec{Q}^1 \right). \tag{14}$$

[1] The author thanks Hans Havlicek for discussion on this issue, as well as supplying a generous amount of references.

Here, $\vec{Q}^i$ denotes the i-th column of Q, and $\beta \in \mathbb{C}$.

However, the latter case is equivalent to the former by way of the reflection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{C})$. Moreover, we note that

$$\left(\vec{Q}^0 \;\; \beta\vec{Q}^0 \right) \sim \left(\vec{Q}^0 \;\; \vec{0} \right) \tag{15}$$

by way of $\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{C})$. As a result, we may always choose representatives of the form $\left(\vec{Q}^0 \;\; \vec{0} \right)$. It is straightforward to then show that the remaining freedom is given by

$$\left(\vec{Q}^0 \;\; \vec{0} \right) \sim \left(\alpha\vec{Q}^0 \;\; \vec{0} \right), \tag{16}$$

for arbitrary non-zero $\alpha \in \mathbb{C}$. Thus, the topological equivalence of the space of singular orbits and $\mathbb{CP}^1 \sim S^2$ is clear. This observation, as well as the non-Hausdorff character of $\mathbb{BP}^0$, were briefly mentioned in [1].

We now consider the the non-singular class p. In $\mathbb{C}^{2\times 2}$, every non-empty open set of $\mathbb{C}^{2\times 2}$ must contain an element of the dense subset $GL(2, \mathbb{C})$. By definition of the quotient topology, every open set of $\mathbb{BP}^0$ must contain p. Thus, p is a dense point. We conclude that $\mathbb{BP}^0$ has the structure of $\mathbb{CP}^1$ with an additional, dense point: $\mathbb{BP}^0 = \mathbb{CP}^1 \cup \{p\}$. This is precisely the structure of the scheme-theoretic version of $\mathbb{CP}^1$, with p playing the role of the *generic point* (see, e.g. [4]).

3. The Twistor Structure of $\mathbb{BP}^0$

3.1. The Basic Twistor Correspondence

As a complex vector space, twistor space[2] $\mathbb{T}$ is isomorphic to $\mathbb{C}^4$:

$$\mathbb{T} \cong \mathbb{C}^4. \tag{17}$$

For current purposes, it is unnecessary to introduce the pseudo-Hermitian form Φ usually considered as part of $\mathbb{T} = (\mathbb{T}, \Phi)$. For $i = 1, 2, 3$, we denote the space of $i-$dimensional subspaces of $\mathbb{T}$ by $\mathbb{F}_i$. For example, we define the Grassmannian of 2-dimensional subspaces of $\mathbb{T}$ by

$$\mathbb{F}_2 := \{\text{span}\{w^\mu, z^\mu\} \mid \{w^\mu, z^\mu\} \text{ is linearly independent}\}. \tag{18}$$

[2] A more thorough treament of the background introduced in this section may be found in [5].

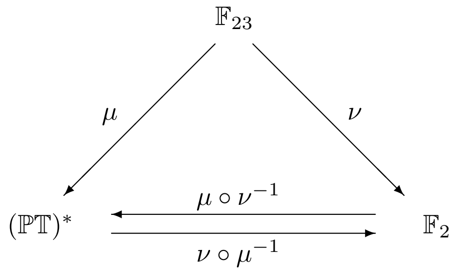
Similarly, projective twistor space $\mathbb{PT}$ is the space of one (complex) dimensional subspaces of $\mathbb{T}$,

$$\mathbb{PT} = \mathbb{F}_1,$$

and the Grassmannian of 3-spaces is $\mathbb{F}_3$. We note that $\mathbb{F}_3$ is naturally isomorphic to the space of dual projective twistors $(\mathbb{PT})^*$. Let $\mathbb{F}_{23} := \mathbb{F}_{23}(\mathbb{T})$ be the flag manifold of two- and three-dimensional nested subspaces of twistor space:

$$\mathbb{F}_{23} = \{(S_2, S_3) | S_2 \in \mathbb{F}_2, S_3 \in \mathbb{F}_3, S_2 \subset S_3\}. \tag{19}$$

The basic (dual) twistor correspondence is then given by the diagram:



The maps μ and ν are the natural projection maps

$$\mu((S_2, S_3)) = S_3, \ \ \nu((S_2, S_3)) = S_2. \tag{20}$$

So far, this is an interesting mathematical construction, but the physical connection is not apparent. To this end, consider four–dimensional complex Minkowski spacetime $\mathbb{CM} := (\mathbb{C}^4, \eta_{\alpha\beta})$, where

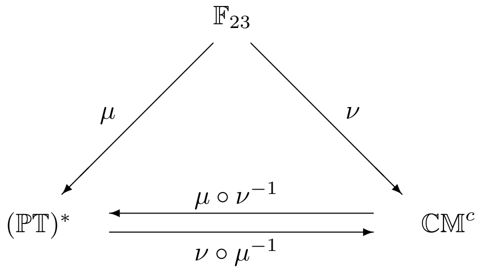
$$(w^0, w^1, w^2, w^3) = w^\alpha \in \mathbb{CM}, \tag{21}$$

$$\eta_{\alpha\beta} w^\alpha w^\beta = (w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2. \tag{22}$$

If we denote the conformal compactification of $\mathbb{CM}$ by $\mathbb{CM}^c$, then it can be shown [5] that

$$\mathbb{F}_2 \cong \mathbb{CM}^c.$$

Thus, we get the more physically relevant diagram



This diagram illustrates the correspondence between (dual, projective) twistors and complexified, compactified spacetime. Of course, there is much more theory (and applications) associated with this type of correspondence to warrant its study than we've alluded to in this brief summary (see, e.g., [5, 6]), however, we will keep to those features pertinent to the subsequent discussion.

3.2. The Twistor Structure of $\mathbb{BP}^0$.

As in [7], we define the location of a (dual) twistor in $\mathbb{CM}$ to be the solutions $x^\alpha \in \mathbb{CM}$ to

$$\tau^{A'} = -ix^{A'A}\sigma_A, \tag{23}$$

where $Z_\alpha = (\sigma_A, \tau^{A'}) \in \mathbb{T}^*$ is fixed (up to scale) and $x^{AA'} = \sigma_\alpha^{AA'} x^\alpha$ is the usual spinor-vector correspondence [6]. The locus of solutions corresponds to $\nu \circ \mu^{-1}([Z_\alpha]) \subset \mathbb{CM}^c$, $[Z_\alpha] \in (\mathbb{PT})^*$ and forms a complex, totally null $2-$plane (i.e. every tangent vector is null and orthogonal to every other tangent vector). Such planes are called $\beta-$ planes. Let $x_0^{AA'}$ be a point of a β-plane. Then, for arbitrary $\lambda^{A'}$, $x^{A'A} = x_0^{A'A} + \lambda^{A'}\sigma^A$ will also lie on the β-plane, and in fact, all points arise in this way.

Choose $x_0^{AA'} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, corresponding to the origin in $\mathbb{CM}$. It follows from (23) that a twistor passing through the origin will have the form $Z_\alpha = (\sigma_A, 0^{A'})$, i.e. $\tau^{A'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So, fix $Z_\alpha = (\sigma_A, 0^{A'})$, $\sigma_A \neq 0$. Then (23) is equivalent to $x^{A'A}$ being singular, so that in matrix terms we must have

$$x^{AA'} = \sigma^A \lambda^{A'}, \tag{24}$$

where we reiterate that σ^A is fixed, yet $\lambda^{A'}$ is varying. Rewrite (24) in columns as

$$\left(\sigma^A \lambda^{0'} \ \ \sigma^A \lambda^{1'} \right). \tag{25}$$

We note that this is precisely the quantities we had in §2.2.2, e.g. equation (13). In other words, the singular orbits in $\mathbb{BP}^0$ correspond to the various β-planes through the origin via:

$$\vec{Q}_0 \leftrightarrow \sigma^A.$$

Consider, instead of the right projective point based on a right scalar action of $GL(2,\mathbb{C})$ on $\mathbb{B}$, the left projective point $\mathbb{BP}^0_{\text{left}}$ (left $\mathbb{B}-$module). Then, as before, we obtain one orbit corresponding to all invertible biquaternions and orbits consisting of subsets of singular matrices. In this case, we choose to express a generic singular matrix in terms of its rows as

$$Q = \begin{pmatrix} \vec{Q}^0 \\ \vec{Q}^1 \end{pmatrix} = \begin{pmatrix} \vec{Q}^0 \\ \beta\vec{Q}^0 \end{pmatrix}. \tag{26}$$

We may similarly show that the orbits for these matrices may be represented by

$$Q = \begin{pmatrix} \vec{Q}^0 \\ \vec{0} \end{pmatrix}, \tag{27}$$

and that $\vec{Q}^0$ is unique up to multiplication by a complex nonzero constant. So, as with the right projective point, we obtain: $\mathbb{BP}^0_{\text{left}} = \mathbb{CP}^1 \cup \{p\}$. So it appears that there is nothing interesting in changing the scalar action, but this is not true! If we proceed to consider the twistor interpretation, we find something very interesting indeed.

The location of a (non-dual) twistor in $\mathbb{CM}$ consists of the solutions $x^\alpha \in \mathbb{CM}$ to

$$\omega^A = ix^{AA'}\pi_{A'}, \tag{28}$$

where $Z^\alpha = (\omega^A, \pi_{A'}) \in \mathbb{T}$. The locus of solutions from this equation forms a complex null 2-plane (i.e. every tangent vector is null and orthogonal to every other tangent vector), called an α-plane. However, it is important to emphasize that α-planes represent (projective) twistors, while β-planes represent dual (projective) twistors. Points on an α-plane passing through the origin in $\mathbb{CM}$ have the form $x^{AA'} = \lambda^A\pi^{A'}$, where λ^A is varying and $\pi^{A'}$ is fixed. In row matrix form we have,

$$\begin{pmatrix} \lambda^0\pi^{A'} \\ \lambda^1\pi^{A'} \end{pmatrix}. \tag{29}$$

Comparing this expression to (26), we see that the elements of $\mathbb{BP}^0_{\text{left}}$ correspond to the $\alpha-$planes passing through the origin in $\mathbb{CM}$. We are thus led to the result that the duality between $\beta-$ and $\alpha-$ planes corresponds, in biquaternion terms, to choosing a right versus a left scalar action (respectively).

4. Concluding Remarks

Recall that either of the projective points $\mathbb{BP}^0$ or $\mathbb{BP}^0_{\text{left}}$ consists of a sphere together with a dense point p. The point p corresponds to those biquaternions that, in turn, correspond to nonsingular matrices, i.e. rank two matrices. For any such matrix, the columns are independent and represent a complex 2-plane. The action of $GL(2,\mathbb{C})$ is then just a change of basis for the plane, and so this plane is represented by the point p.

In light of the results in §3.2, together with the representation of compactified, complexified Minkowski space, $\mathbb{CM}^c$, in terms of the Grassmannian $\mathbb{F}_2$, we should identify the point p with the origin in $\mathbb{CM}^c$. The biquaternionic projective points then have a complete twistorial interpretation as a point (the origin) of $\mathbb{CM}^c$ whose closure adds the $\alpha-$ or $\beta-$ planes associated with the point.

In a forthcoming paper, we describe, among other things, how these ideas can be combined with the results of Souček to obtain a description of $\mathbb{BP}^1$ as encompassing all points of $\mathbb{CM}^c$ and their corresponding $\alpha-$ or $\beta-$ planes in the manner described in this paper. Generalizations of these constructions to higher dimensions are discussed as well. The possible role of schemes in the higher dimensional cases is also being pursued, though the issue is currently less clear.

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